Exercise 11M – Worked Solutions to the Problems

Hints

1. Draw a neat sketch.

   Here is our plan:
   1. Find gradient of tangent at P.
   2. Find gradient of normal through P.
   3. Find equation of normal through P.
   4. Find *x*-coordinate of Q by solving linear and quadratic equations simultaneously.
   5. Find *y*-coordinate of Q by substituting and evaluating.
   6. Find gradient of tangent at Q.
   7. Find equation of tangent at Q.

2. Solving this problem is largely a case of "following your nose.". First find the 1st and 2nd derivatives and then use algebra to show that the RHS equals the LHS.

3. Start with a neat sketch. Then:
   1. Find the *x*-coordinates of the points of intersection of the two curves.
   2. For each curve, find the gradient and hence angle between the positive x-axis and the tangent lines at each point.
   3. Subtract to find the angle between the tangent lines.

4. Find the coordinates of the point where the rocket is launched. Then find the gradient of the line between this point and the space station. Find the gradient of the tangent line from (27 000, 18 900). Are these gradients equal? What does this mean?

5. Note: there is a typo in the question. The lowercase 'p' in the equation should be an uppercase 'P'. Write the equation in terms of P and differentiate. Now here is the tricky bit. You need to re-write the RHS so it contains the expression $kv^{-1.5}$. Then use algebra to get the desired expression.

6. Write with a negative index, and differentiate. Recall that the quantities m_1 and m_2 are constants.

7. Write in index form, differentiate, and re-write in root notation.

8. Use power regression on a graphics calculator to find the function. The differentiate, substitute and evalulate.

9. For each problem, write in index form, differentiate and simplify.

10. A general proof is too difficult at this stage. Prove it for functions of the form $y = x^n$.
    Recall that an even function of the form $y = x^n$ has an index that is an even number (say of the form 2*k*), while an odd function of the form $y = x^n$ has an index that is an odd number (say of the form 2*k*+1).

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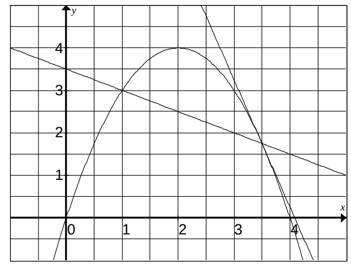


1. A neat sketch is needed. See alongside.

Here is our plan:

1. Find gradient of tangent at P.
2. Find gradient of normal through P.
3. Find equation of normal through P.
4. Find *x*-coordinate of Q by solving linear and quadratic equations simultaneously.
5. Find *y*-coordinate of Q by substituting and evaluating.
6. Find gradient of tangent at Q.
7. Find equation of tangent at Q.

Step 1

Expanding gives: $y = 4x - x^2$.

Differentiate: $\dfrac{dy}{dx} = 4 - 2x$

At $x = 1$ $\quad m = \dfrac{dy}{dx} = 4 - 2(1) = 2$

Step 2

Gradient of normal is the negative reciprocal of the gradient of the tangent, so

$$m_N = -\frac{1}{2}$$

Step 3

Using $y - y_1 = m(x - x_1)$

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$y - 3 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{7}{2}$$

Step 4

Set expressions equal to each other, and solve for *x*.

$$4x - x^2 = -\frac{1}{2}x + \frac{7}{2}$$

$$x^2 - \frac{9}{2}x + \frac{7}{2} = 0$$

$$2x^2 - 9x + 7 = 0$$

$$(2x - 7(x - 1) = 0$$

$$x = 1, \quad x = \frac{7}{2} = 3\frac{1}{2}$$

Step 5

$$y = \frac{7}{2} \times \frac{1}{2} = \frac{7}{4} = 1\frac{3}{4}$$

Therefore coordinates of Q are $(3\frac{1}{2},\ 1\frac{3}{4})$

Step 6

$$m = \frac{dy}{dx} = 4 - 2x$$

$$m_Q = 4 - 2(\frac{7}{2}) = 4 - 7 = 3$$

Step 7

$$y - y_1 = m(x - x_1)$$

$$y - \frac{7}{4} = -3(x - \frac{7}{2})$$

$$y - \frac{7}{4} = -3x - \frac{21}{2}$$

$$y = -3x - \frac{42}{4} + \frac{7}{4}$$

$$y = -3x - \frac{49}{4}$$

$$y = -3x - 12\frac{1}{4}$$

2. Solving this problem is largely a case of "following your nose.". Find the 1st and 2nd derivatives:

$$f(x) = ax + bx^{-1}$$

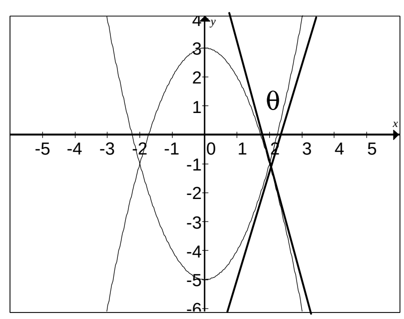
$$f'(x) = a - bx^{-2} = a - \frac{b}{x^2}$$

$$f''(x) = 2bx^{-3} = \frac{2b}{x^3}$$

Use algebra to show that the right-hand side (RHS) equals the left-hand side (LHS).

$$\begin{aligned} RHS &= x\ f'(x) + x^2\ f''(x) \\ &= x(a - \frac{b}{x^2}) + x^2(\frac{2b}{x^3}) \\ &= ax - \frac{b}{x} + \frac{2b}{x} \\ &= ax + \frac{b}{x} \\ &= f(x) \\ &= LHS \end{aligned}$$

3. Start with a neat sketch. Then:
   1. Find the x-coordinates of the points of intersection of the two curves.
   2. For each curve, find the gradient and hence angle between the positive x-axis and the tangent lines at each point.
   3. Subtract to find the angle between the tangent lines.



Step 1

$$-x^2 + 3 = x^2 - 5$$

$$2x^2 - 8 = 0$$

$$x^2 - 4 = 0$$

Therefore $x = 2, \quad x = -2$

Step 2

Gradient of $f(x)$ at $x = 2$:

$$f(x) = -x^2 + 3$$
$$m = f'(x) = -2x$$

At $x = 2$, $\quad m = f'(2) = -2\times 2 = -4$

Now find the angle this makes with the positive x-axis:

$$\begin{aligned} q &= \tan^{-1}(x) \\ &= \tan^{-1}(-4) \\ &= -75.96^0 \\ &= -75.96^0 + 180 \\ &= 104.94^0 \end{aligned}$$

Gradient of $g(x)$ at x =2:

$$g(x) = x^2 - 5$$
$$m = g'(x) = 2x$$

At $x = 2$, $\quad m = g'(2) = 2\times 2 = 4$

Now find the angle this makes with the positive x-axis:

$$\begin{aligned} q &= \tan^{-1}(x) \\ &= \tan^{-1}(4) \\ &= 75.96^0 \end{aligned}$$

Step 3

Find the angle between these lines:

$$q = 104.04^0 - 75.96^0 = 28.08^0$$

By the symmetry of the graph, the angle between the tangent lines at $x = 2$ is also 28.08^0.

4. Find the coordinates of the point where the rocket is launched:

At $x = 27\,000$, $\quad y = 27\,000^{\frac{2}{3}} + \dfrac{2\times 27\,000}{3}$

$$= 18\,900$$

The rocket was launched at (27 000, 18 900).

Find the gradient of the line between this point and the space station:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1.3\times 10^5 - 18\,900}{93\,300 - 27\,000} \\ &= 1.6 \end{aligned}$$

Find the gradient of the tangent from (27 000, 18 900):

$$\begin{aligned} \frac{dy}{dx} &= \frac{2}{3}d^{-\frac{1}{3}} + \frac{2}{3} \\ &= \frac{2}{3}\times 27\,000^{-\frac{1}{3}} + \frac{2}{3} \\ &= 0.689 \end{aligned}$$

Since the gradients are not equal, the rocket does not lay a course for the space station.

5. Note: there is a typo in the question. The lowercase ‘p’ in the equation should be an uppercase ‘P’.

Write the equation in terms of P:

$$Pv^{1.5} = k$$

$$P = \frac{k}{v^{1.5}}$$

$$P = kv^{-1.5}$$

Differentiate:

$$\frac{dP}{dv} = -1.5kv^{-2.5}$$

Now here is the tricky bit. We need to re-write the RHS so it contains the expression $kv^{-1.5}$.

Using index laws:

$$\frac{dP}{dv} = -1.5kv^{-2.5}$$

$$= -1.5kv^{-1.5}v^{-1} \qquad \{\text{index law 1}\}$$

$$= \frac{-1.5kv^{-1.5}}{v} \qquad \{ v^{-1} = \frac{1}{v} \}$$

$$= \frac{-1.5P}{v} \qquad \{\text{since } P = kv^{-1.5} \}$$

Therefore $\frac{dP}{dv} = \frac{-1.5P}{v}$.

6. Write with a negative index:

$$F = Gm_1m_2r^{-2}$$

The quantities m_1 and m_2 are constants, so:

$$\frac{dF}{dr} = -2Gm_1m_2r^{-3}$$

$$= \frac{-2Gm_1m_2}{r^3}$$

7. Write in index form:

$$T = 2p\sqrt{\frac{l}{g}}$$

$$= 2p\sqrt{\frac{1}{g}}\sqrt{l}$$

$$= \frac{2p}{\sqrt{g}}l^{\frac{1}{2}}$$

Differentiate:

$$\frac{dT}{dl} = \frac{1}{2}\frac{p}{\sqrt{g}}l^{-\frac{1}{2}}$$

$$= \frac{p}{2\sqrt{g}\,l^{\frac{1}{2}}}$$

$$= \frac{p}{2\sqrt{gl}}$$

Therefore $\frac{dT}{dl} = \frac{p}{2\sqrt{gl}}$

8. Using power regression on a graphics calculator, we find that the function is:

$$y = 3x^{1.5}$$

Find the derivative:

$$\frac{dy}{dx} = 4.5x^{0.5}$$

At $x = 2.5$, $\frac{dy}{dx} = 4.5 \times 2.5^{0.5}$

$$\approx 7.11$$

Therefore $\frac{dy}{dx} \approx 7.11$

9. For each problem, write in index form, differentiate and simplify:

Given: $f = \frac{1}{2L}\sqrt{\frac{T}{r}}$

a. $f = \frac{1}{2}\sqrt{\frac{T}{r}}\ L^{-1}$

$$\frac{df}{dL} = -\frac{1}{2}\sqrt{\frac{T}{r}}\ L^{-2}$$

$$= \frac{-1}{2L^2}\sqrt{\frac{T}{r}}$$

b. $f = \frac{1}{2L\sqrt{r}}\sqrt{T}$

$$= \frac{1}{2L\sqrt{r}}\ T^{\frac{1}{2}}$$

$$\frac{df}{dT} = \frac{1}{2L\sqrt{r}}\sqrt{T}$$

$$= \frac{1}{2}\frac{1}{2L\sqrt{r}}\ T^{-\frac{1}{2}}$$

$$= \frac{1}{4L\sqrt{r}\sqrt{T}}$$

$$= \frac{1}{4L\sqrt{rT}}$$

c. $f = \frac{\sqrt{T}}{2L}\ r^{-\frac{1}{2}}$

$$\frac{df}{dr} = -\frac{1}{2}\frac{\sqrt{T}}{2L}\ r^{-\frac{3}{2}}$$

$$= \frac{-\sqrt{T}}{4L\sqrt{r^3}}$$

This answer is ok, but it can be better expressed as follows:

$$\frac{df}{dr} = \frac{-\sqrt{T}}{4L\sqrt{r^3}}$$
$$= \frac{-\sqrt{T}}{4L\sqrt{r^2}\sqrt{r}}$$
$$= \frac{-1}{4Lr}\sqrt{\frac{T}{r}}$$

10. A general proof is too difficult at this stage. We will prove it for functions of the form $y = x^n$. Recall that an even function of the form $y = x^n$ has an index that is an even number, while an odd function of the form $y = x^n$ has an index that is an odd number.

If n is an even number then we can write $n = 2k$, where k is an integer. Hence:

$$y = x^{2k}$$
$$\frac{dy}{dx} = 2x^{2k-1}$$

This is a power function with an odd index, hence the derivative is an odd function.

If n is an odd number then we can write $n = 2k+1$, where k is an integer. Hence:

$$y = x^{2k+1}$$
$$\frac{dy}{dx} = 2x^{2k}$$

This is a power function with an even index, hence the derivative is an odd function.