Chapter 1 – Worked Solutions to Exercise 1J

Learning to solve problems

Like any worthwhile activity, learning to solve mathematics problems is a skill that takes time to develop. Here are some ideas on how to become a great problem solver:

Read each problem carefully	Make sure that you know exactly what information is given, and what you are being asked to find. Often it helps to read the question a few times.
What mathematic topics may be useful?	Problems given in textbooks are based on mathematics that you have learned – either in that chapter, or previously. If you can figure out which mathematics you need to tackle the problem, you are more than half way to its solution.
Revise the examples	Often the mathematics that is needed has been introduced is in the current chapter. Work through the examples of the mathematical procedures that you think will be needed to solve the problem.
Use the hints	At this early stage of learning to solve problems, you may find even after reading the problem and working the examples, you are still not sure how to proceed. At this stage, read the hints, slowly and carefully, and then go back and have another go at the problem.
Look at the solution	It is likely that there will be problems that you are unable to solve, especially the challenging problems at the end of a problem set. For these, read through the solution, then put it aside and try to set out the solution on your own. If you get stuck, look at the solution again. Repeat this until you can do the solution unaided.
Do *all* of the problems	Any complex activity, such as hitting a golf ball or solving a problem, requires practice. You will become a good problem solver by solving lots of problems!
Be persistent	By definition, a problem is a question for which the method of solution is not immediately obvious. If you can't solve a problem after a good attempt, set it aside and come back to it later. Often you will find that you will see a method of solving the problem when you return to it after a break.

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Hints

1. We need to understand about what $-h(-a)$ actually means. For example, consider $h(x) = x^3$ where $x = 2$.
   What is the value of $h(2)$?
   What is the value of $h(-2)$?
   What is the value of $-h(-2)$?
   Repeat for other values.

   Draw the graph of $h(x) = x^3$. Graphically, explain what is meant by $-h(-a)$?

   On your graphics calculator, graph $Y1 = x^3$ and $Y2 = -Y1(-x)$. Are they the same?

   Change Y1 to another simple function. Are Y1 and Y2 still the same? Explore more functions. For which ones does this property hold? What do they have in common?

2. What about the horizontal line through (2,1)? The vertical line? That's two lines through (2,1)!

   What information do you need to find the equation of a line? What information is already given? What do you have to supply yourself?

3. On graph paper, carefully plot the given functions. One of them isn't quite right – which one?

4. What information do you need to find the equation of a line? Can you get that information from the graph?

5. What is the domain of this function?

   Make a table of values.

6. If you know the equation of a function and the *x*-coordinate, how do you find the *y*-coordinate?

   What do you know about the gradient of perpendicular lines?

7. Draw an accurate diagram!

   What do you know about the gradient of parallel lines?

   List EVERYTHING you know about these lines. Have you used ALL of this information?

   How many lines are there that are parallel to a given line, and at a distance of 5 units?

   This is a challenging problem! You will learn a great deal of mathematics, and receive much satisfaction, if you persist until you solve it, even if it takes weeks (and some problems do that that long or longer to solve!)

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Solutions

1. *Model solution – there are other approaches to this problem.*

It is easy to show that this is not true for all functions. For example, let $h(x) = x + 3$. Then $h(2) = 2 + 3 = 5$, while $-h(-2) = -(-2 + 3) = -1$.

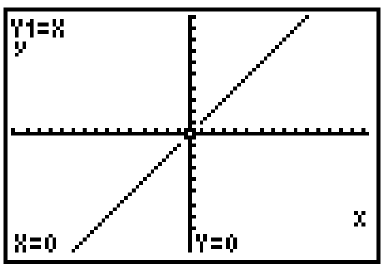
An investigation will show that there are other functions for which this pattern holds true:

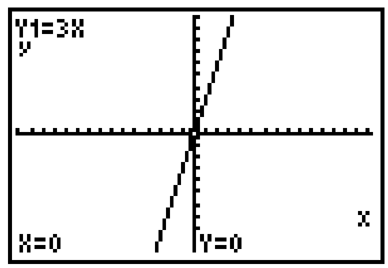
$y = x$ $\qquad$ $y = 3x$

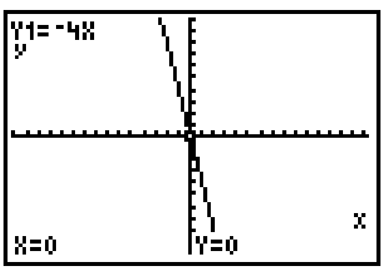
$y = -4x$ $\qquad$ $y = 2x^3$

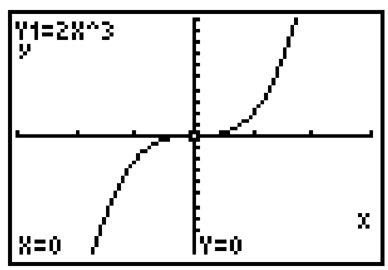
$y = x^3 + 2x$ $\qquad$ $y = x^5 - 2x^3 + 3x$

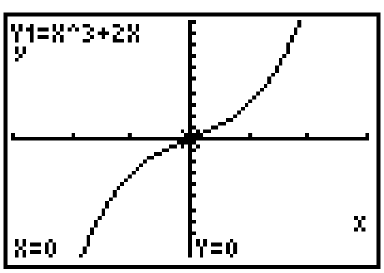
Here are the graphs of these functions.

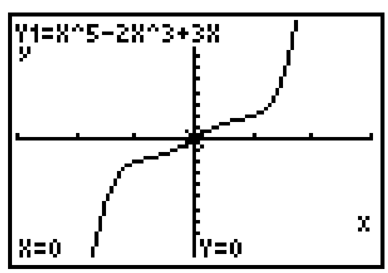












What do these functions have in common?

they only contain only powers of *x* that are positive odd numbers.

their graphs are all symmetric about the origin – the "top" half of the graph, rotated by 180^0, gives the bottom half of the graph.

I conclude that this identity holds true only for graphs that have the above properties.

N.B. This conclusion turns out to be false! But it is still a model solution for a student just starting Mathematics B.

2. There is not a single correct answer to this problem.

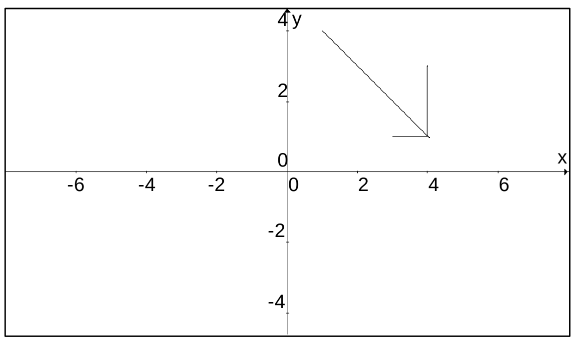
Two lines are easy to find: the horizontal line through (2,1) has the equation $y = 1$, while the horizontal line through (2,1) has the equation $x = 2$.

To find other lines, we need to know either the coordinates of two points, or the coordinates of one point and the gradient. We know the coordinates of one point, so we have to supply either a second point or a gradient. The gradient is easier, as we can then use $y - y_1 = m(x - x_1)$.

If *m* = 1, then we have: $y - 1 = 1(x - 2) \quad \rightarrow \quad y = x - 1$
If *m* = 2, then we have: $y - 1 = 2(x - 2) \quad \rightarrow \quad y = 2x - 3$
If *m* = 3, then we have: $y - 1 = 3(x - 2) \quad \rightarrow \quad y = 3x - 5$

Substituting 2 for *x* and 1 for *y* into these functions confirms that these are solutions to the problem.

3. A careful plot of the functions yields the following graphs:



It is obvious that the vertical part of the arrowhead is too long. She needs to change the range to $1 \le y \le 2$.

4. From the graph, it is possible to determine the gradient and *y*-intercept, and substitute these into the linear function $y = mx + c$:

main arrow: gradient = –1, y-intercept = 2 → $y = -x + 2, \quad -1 \le x \le 4$

lower arrowhead: gradient = -½ , y-intercept = 0 → $y = -\frac{1}{2}x, \quad 2 \le x \le 4$

upper arrowhead: gradient = -2 , y-intercept = 6 → $y = -2x + 6, \quad 3 \le x \le 4$

5. Construct a table of values, and determine the equation from the table.

x (hours worked)	*y* (total daily earnings)
8	160
10	220
12	280

Choosing any two of these pairs of values and doing the necessary algebra gives the function:

$$y = 30x - 80, \quad 8 \le x \le 12.$$

Substitution confirms that this is the correct equation.

There are other ways of thinking about this problem that arrive at the same answer (or an equivalent form of the answer). It would be instructive for a class to share their methods of solution to this problem.

6. This is what we call a "follow your nose" problem – there is essentially one path that leads to the correct answer. The path for this question requires that we first find y, then x, and then the coordinates of B.

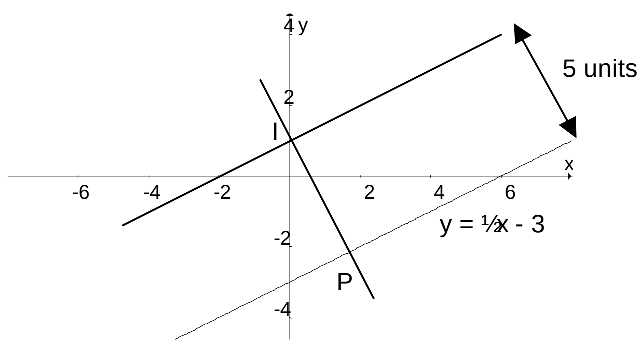
To find y: Substitute $x = 1$ into the function $y = 2x + 2$. This gives the coordinates (1, 4).

To find x: First, we need the equation of the line. We know the coordinates of a point on the line are (1,4) and the gradient is $-\frac{1}{2}$ (i.e. the negative reciprocal of 2). A bit of algebra yields the equation $y = -\frac{1}{2}x + \frac{9}{2}$. Substituting 1 for y, and solving gives $x = 7$.

To find B: The x-coordinate is that found above, i.e. $x = 7$. The y-coordinate is 0. The coordinates of B are (7,0).

7. The given line has a y-intercept of –3 and a gradient of $\frac{1}{2}$. The line we are to find also has a gradient of $\frac{1}{2}$ (since it is parallel to the given line), so what we need to find is its y-intercept.

Here is a sketch of the problem.



Since the lines are parallel we know that the line we have to find has equation $y = \frac{1}{2}x + c$ so our problem is to find c, the y-intercept. The coordinates of I are $(0,c)$

In the diagram the line from I to P is the perpendicular from I and c is to be found so that the length of IP = 5.

Now the gradient of IP is -2 so it has equation $y = -2x + c$. The value of c is the same, because in the equation above it is the y intercept in this second equation.

Now the coordinates of P are the values where the lines $y = -2x + c$ and $y = \frac{x}{2} - 3$, intersect.

Solving these equations, equating the expressions for y we have

$$\frac{x}{2} - 3 = -2x + c.$$

Thus we have $\frac{5x}{2} = c + 3$, so $x = \frac{2c}{5} + \frac{6}{5}$. Substituting into one of these equations to find y we have

$$y = -2\left(\frac{2c}{5} + \frac{6}{5}\right) + c = \frac{-4c}{5} - \frac{12}{5} + c = \frac{c}{5} - \frac{12}{5}.$$

The coordinates of P are thus $\left(\left(\frac{2c}{5} + \frac{6}{5}\right), \left(\frac{c}{5} - \frac{12}{5}\right) \right)$ and the coordinates of I are (0, c).

The length of IP is 5 so applying the distance formula using the coordinates of I and P we have

$$\left(\left(\frac{2c}{5} + \frac{6}{5}\right) - 0 \right)^2 + \left(\left(\frac{c}{5} - \frac{12}{5}\right) - c \right)^2 = 25$$

which simplifies to

$$\left(\frac{2c}{5} + \frac{6}{5}\right)^2 + \left(-\frac{12}{5} - \frac{4c}{5}\right)^2 = 25,$$

and further simplifies to

$$\frac{4c^2 + 24c + 36}{25} + \frac{144 + 96c + 16c^2}{25} = 25$$

Further rearrangement gives

$$20c^2 + 120c + 180 = 25^2$$

Cancelling the factor 5 results in

$$4c^2 + 24c - 89 = 0.$$

Solving this quadratic for c gives two solutions,

$$c = \frac{-24 \pm \sqrt{24^2 + 4 \times 4 \times 89}}{8} = \frac{-24 \pm \sqrt{2000}}{8} = \frac{-24 \pm 20\sqrt{5}}{8} = -3 \pm \frac{5\sqrt{5}}{2}.$$

We didn't mention 2 solutions, but when we think about the problem we could have sketched a line under the given line in the diagram, this too would be 5 units from the given line. So there are two parallel lines distance 5 units from the line $y = \frac{1}{2}x - 3$. They are:

$$y = -2x - 3 - \frac{5\sqrt{5}}{2} \quad \text{and} \quad y = -2x - 3 + \frac{5\sqrt{5}}{2}$$