Chapter 2 - Worked Solutions to Exercise 2I

Hints

1. a. If you multiply a number by 1 you don't change its value. And the number 1 can be expressed in many ways, e.g. $\frac{2}{2}, \frac{-3}{-3}, \frac{\sqrt{2}}{\sqrt{2}}$.
   b. hint is already given in the question.
   c. see previous hint!

2. We can add like surds, e.g. $\sqrt{3}+2\sqrt{3}=3\sqrt{3}$. Can you re-write this expression so it has the form $\sqrt{a}+\sqrt{b}=\sqrt{c}$?

3. The simplest quadratic function is $y=x^2$. This passes through (1,1). What transformation results in a quadratic function that passes through (1,2)?

4. What is the link between the roots of a quadratic function and its factored form? What is the effect on the roots of a function if the function is stretched vertically by a factor of *a*?

5. What is the inverse operation of taking the square root?

6. Simplify both sides first.

7. Let x be the number (we have to call it something). What is the square of the number? Write the algebraic expression for "the amount by which a number exceeds its square". Can you write this as a function? How can you find the maximum value of this function?

8. What are the coordinates of the vertex? Translate $y=x^2$ so the coordinates of the vertex are correct. Now what other transformation is needed so the graph passes through the other points?

9. You will need a ruler here! You can choose where to put the origin, and some choices are better than others.

10. A hint is given along with the question. A further hint is to then isolate the term containing the square root on one side of the equation.

11. What do the coordinates of B represent? The coordinates of A? Can you use the answers to these questions to write an expression for the length of BA?

12. What is the expression for the *total* weight gain for *n* fish, if each fish gains $60 - 25n$ grams?

13. Where is the origin located? What is the equation of a line that passes through the origin and is 45^0 to the horizontal?

14. Draw the diagram! What are your two variables? Add them to the diagram. How will you use the fact that there is a total of 3600 m of fencing? Can you express one variable in terms of the other?

15. There are two variables, *x* and *y*. Can you express one of them in terms of the other?

16. A common method of locating the incorrect step is to substitute values for *a* and *b* and work through the argument.

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Solutions

1. a. Here are a few solutions, found by multiplying $\sqrt{2}$ by 1, in its many guises:

$$\sqrt{2}\times\frac{2}{2}=\frac{2\sqrt{2}}{2}=\frac{\sqrt{8}}{2},\ \sqrt{2}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{6}}{\sqrt{3}}.$$ Use this method to find three others.

Check your solutions with your calculator.

b. Square both sides: $(5\sqrt{6})^2=25\times 6=150$; $(6\sqrt{5})^2=36\times 5=180$. Since the square of $6\sqrt{5}$ is greater than the square of $5\sqrt{6}$, then $6\sqrt{5}$ is greater than $5\sqrt{6}$.

c. $(m\sqrt{n})^2>(n\sqrt{m})^2$ {square both sides}

$m^2n>n^2m$ {expand}

$\dfrac{m^2n}{mn}>\dfrac{n^2m}{mn}$ {divide both sides by mn}

$m>n$

It must be true that $m > n$.

2. Using the hint,

$$\sqrt{3}+2\sqrt{3}=3\sqrt{3} \text{ can be written as } \sqrt{3}+\sqrt{12}=\sqrt{27}.$$

We can find an infinite number of solutions by starting with a sum or difference involving like surds. Make up three more of your own.

3. The simplest quadratic is $y=x^2$. When $x = 1$, $y = 1$. If we apply a vertical stretch by a factor of 2, the function will pass through (1,2). The function is $y=2x^2$.

4. The quadratic function $y=(x+3)(x-2)$ has zeros at x = -3 and x = 2. We can find other quadratic functions with the same zeros by applying a vertical stretch, as this doesn't affect the zeros. Hence, $y=2(x+3)(x-2)$ is another function with zeros at (-3,0) and (2,0).

5. If $\sqrt{\sqrt{\sqrt{x}}}=2$, then we can square both sides to get $\sqrt{\sqrt{x}}=4$. Squaring two more times gives $x=((4)^2)^2=16^2=256$.

6. Quite often you can see special solutions of equations. You can always try $x = 0$ and $x = 1$.
$x = 0$ fits since $\sqrt{0}=0$ but since $\sqrt{1}=1$ on the left we get 3 and on the right we get 1, so $x = 1$ is not a solution. So we can proceed knowing we have a solution already.
Collecting the three square roots on the left we see that the equation is just $3\sqrt{x}=\sqrt{x}\sqrt{x}\sqrt{x}$
We can cancel the factor $\sqrt{x}$ noting that $\sqrt{x}=0$ is a possible solution, which we have already noted in any case. Then we have $3=\sqrt{x}\sqrt{x}=x$.

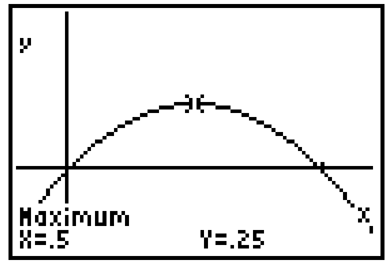
Thus we have found the two solutions $x = 0$ and $x = 3$.

7. The number concerned has to have a name, so call it x. The question is about the number itself and its square. Thus the question concerns x and x^2 .

It asks that we find x so that 'x exceeds x^2 by the greatest amount'.

The amount by which x exceeds x^2 is given by $x - x^2$.

We have to find the value of x for which this quantity is greatest. In other words we want the maximum value of $x - x^2$.



Set Y1 of your graphics calculator equal to this expression and graph it. Find the coordinates of the maximum value using the 'maximum' menu item. From the screen shot alongside, the maximum value of ¼ occurs at x = ½.

8. By symmetry, it is obvious that the graph has its vertex at (-1 –3), so the equation is of the form $y = A(x+1)^2 - 3$. To find A, we substitute the coordinates of one point:

$$3 = A(1+1)^2 - 3$$
$$3 = 4A - 3$$
$$A = \frac{3}{2}$$

Hence the equation of the parabola is $y = \frac{3}{2}(x+1)^2 - 3$.

9. I would choose the centre of the arch to correspond to $x = 0$, and make the roadway the x-axis.

The equation of the arch would then be $y = dx^2 + h$. According to the scale the height of the arch is approximately 1000(2 cm.) = 20 metres. Thus $h = 20$. the width of the bridge is 1000(13 cm.) = 130 metres. so each end of the bridge is at $x =$ 650 metres and − 650 metres. since the height is zero at x = 650 metres, we have $0 = d\,(650)^2 + 20$. Thus $d = -20/(650)^2$ = −1/21125 = − .0000473 to 3 places.

The equation of the arch is thus $y = -\frac{x^2}{21125} + 20$.

10. Squaring both sides:

$$x^2 - x = (2\sqrt{x} - x)^2$$
$$x^2 - x = 4x - 4\sqrt{x}x + x^2$$

Thus $$-x = 4x - 4x\sqrt{x},$$
$$-5x = -4x\sqrt{x}$$

There is a factor x which can be cancelled after noting that $x = 0$ is a possible solution.

Then $\sqrt{x} = \frac{5}{4}$ so $x = \frac{25}{16}$.

Now we have squared both sides, twice and on each occasion it may introduce solutions which are not solutions of the original problem. So we must check out $x = 0$ and $x = 25/16$. In fact both values do fit the original equation.

The solutions are $x = 0$ and $x = \frac{25}{16}$.

11. The length BA is equal to the *x*-coordinate of B – the *x*-coordinate of A, or algebraically,

$$L = (x+4) - 0.5x^2$$
$$L = -0.5x^2 - x + 4$$

The vertex of the graph is a maximum since the coefficient on x^2 is negative. The x-coordinate of the vertex is found using:

$$x = \frac{-b}{2a} = \frac{1}{-1} = -1$$

The maximum value is found by substitution:

$$L = -0.5(1^2) - 1 + 4$$
$$L = 4.5$$

The maximum length of AB is 4.5 units.

12. The total weight of the fish = (number of fish) x (weight per fish). Algebraically,

$$W = n(60 - 25n)$$
$$W = -25n^2 + 60n$$

The number of fish per cubic metre that maximises the total weight is given by:

$$n = \frac{-b}{2a} = \frac{-60}{-50} = 1.2$$

Can we *have* 1.2 fish – shouldn't this answer be a positive integer? Well, think of this number as being an average. For example, if the dam has a volume of 100 m^3, then the farmer should stock 120 fish.

13. The coordinate system has the x axis running horizontally at ground level and the y axis passes through the apex, or top of the monument. If you let the right hand end of the support be at $x = a$ then from the diagram and the information that the angle marked is 45°, the value of y at that point is a. (Since the angle is 45° the line through the right hand end point is $y = x$.)

So the point (a,a) is on the parabola, thus we must have $a = 15 - \dfrac{a^2}{p}$.

Rearranging it we have $ap = 15p - a^2$ that is $a^2 + ap - 15p = 0$

The solutions of this quadratic equation are $a = \dfrac{-p \pm \sqrt{p^2 - 4(1)(-15p)}}{2} = \dfrac{-p \pm \sqrt{p^2 + 60p}}{2}$

We only want the positive value so we have $a = \dfrac{-p + \sqrt{p^2 + 60p}}{2}$. By symmetry the length of the beam is $2a$, thus the length of the beam is $-p + \sqrt{p^2 + 60p}$ metres $\simeq 10.94$ metres.

14. If the rectangular paddocks measure l metres long and w metres wide then we have to find l and w. The total amount of fencing consists of 6 long sides and 8 short sides. Since l is the long side and w the short side, the total amount of fencing needed equals $6l + 8w$. We thus have

$$6l + 8w = 3600 \qquad (1)$$

Since there are 6 rectangles the total area A is given by

$$A = 6lw. \qquad (2)$$

We can eliminate either l or w. Since $6l$ is given in (1) we eliminate l, so $6l = 3600 - 8w$
Thus we get $A = (3600 - 8w)8w$. Multiplying out we have

$$A = 28800w - 64w^2$$

The graph of A is concave down, so there is a maximum when

$$w = \frac{-b}{2a} = \frac{-28800}{-128} = 225$$

Substituting into (1) and solving gives $l = 300$,
and from (2), $A = 405000$ m^2.

15. The width of the metal will be the sum of the lengths shown in the diagram of the cross section, so

$$W = 3 + x + y + x + 2$$

$$W = 2x + y + 5 \qquad (1)$$

Now the area has to be 60 cm^2, and x and y are the lengths of the sides of the area, so

$$xy = 60$$

$$y = \frac{60}{x} \qquad (2)$$

From (2) $y = \frac{60}{x}$, thus (1) becomes

$$W = 2x + \frac{60}{x} + 5 \qquad (3)$$

Using a graphics calculator, we find that there the minimum width of the metal of 26.91 cm occurs when x =5.48 cm.

16. Here is the alleged logical argument, with reasons:

$a = b$	{we assume this to be true}
$a^2 = ab$	{multiply both sides by a}
$a^2 - b^2 = ab - b^2$	{subtract b^2 from both sides}
$(a + b)(a - b) = b(a - b)$	{factorise each side of the equation}
$a + b = b$	{divide both sides by $a - b$}
$2b = b$	{since $a = b$}
$2 = 1$	{divide both sides by b}

An effective method of finding a flaw in such reasoning is to substitute values for a and b, and see where the argument breaks down. For example, set $a = b = 2$ and follow the above steps:

$2 = 2$	{true!}
$4 = 4$	{still true}
$0 = 0$	{still true, but looking suspicious}
$0 = 0$	{still true}
$\frac{0}{0} = \frac{0}{0}$	{illegal! since $a = b$, by dividing by $a - b$, we are dividing by 0}

The expression $\frac{0}{0}$ is called **indeterminate**, and is not a real number. You will learn more this indetermine form in Chapter 11 of your textbook.

This problem illustrates the danger of dividing by zero – we can arrive at an incorrect solution.