# Chapter 5 – Worked Solutions to the Problems

## Hints

1. Start by writing a formula for the area of a triangle. Note that there are two possible angles. Now knowing the angle and two sides, find the third side - which rule applies here?

2. Draw a diagram – make sure that you have it right. Join A and M, and join E and M. Now there are a lot of right angles and right-angled triangles. Introduce one more by joining M to the side on the opposite side of the square. Now everything can be done using trigonometric ratios.

3. Draw a map of the pilot's trip. Have the direction of north pointing towards the top of the page. Draw in the intended route - the one he should have taken. Draw in the new path he needs to take. You will have a triangle.

4. Again draw a diagram showing where the two boats are when they are on their voyages. Calculate and mark the position of both at 6 pm in the afternoon. Draw in the vector from one boat to the other. Calculate the length of the vector and check if it is long enough.

5. a. Put C equal to an angle in which C is the right angle in a right angled triangle
   b. Draw in the perpendicular from B, it has length $h$. Relate $h$, $a$ and C.

6. Use the diagram in question 5 but with the perpendicular coming down from C. Then apply the definition of $\cos A$ and $\cos B$ and you'll find 2 lengths which together make the side $c$. Now in fact if $A$ or $B$ is obtuse (between $90°$ or $180°$) then there's a subtraction involved, because although the plus sign remains, either $\cos A$ or $\cos B$ is negative. You can see this by drawing a triangle with $B$ or $A$ obtuse.

7. Join the centre of the circle to each of the vertices of the hexagon. You have 6 triangles. Are they identical? Do they have a property for which we have a special name? Find their area, and then use a little algebra finishes it off.

8. Draw a "side-on" diagram with the 2 metres vertical segment representing the position of the picture and another point 1 metre below corresponding to the level of the eye of the person. Then draw lines from the person's eye position to the top and bottom of the picture segment. These lines make angles with the horizontal line from the picture to the eye, the angle between them having to be 20 °. Using the definition of the tangent of an angle you can express the horizontal distance in two ways each involving an unknown angle. So an equation for the angle may be found. Find an approximate solution of this equation, and the distance from the picture can be calculated from this.

9. a. The sum of the two small triangles areas equals the area of the large one. Use this and apply some simple algebra.
   b. The ratios in part (a) are cosines.

10. You have an equilateral triangle and segments with angle 60 ° at the centre. Express the given area in terms of these elements and do some algebra to get the required formula.

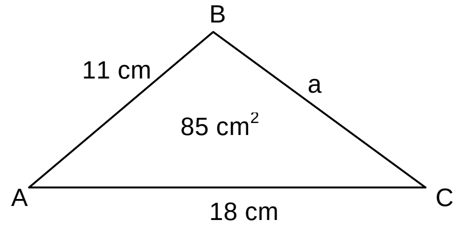
11. You need to remember the geometric properties of circles, tangents, radii and parallel lines. The length required is the sum of four pieces. There is a θ missing from the formula for belt length, it should be:

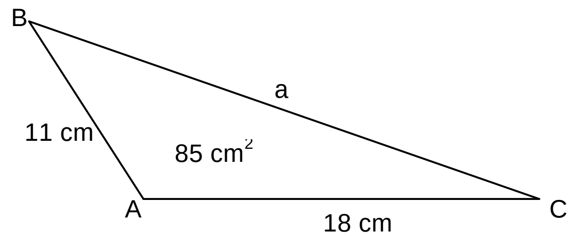
$$\text{belt length} = p\,(a+b) + 2q\,(a-b) + 2c\cos q \;.$$

12. The diagram for this problem is really the same as that used in problem 8 and the same formulae may be used. The manipulations and the objective are different but the basis of the problem is the same.

# Chapter 5 – Worked Solutions to the Problems

1. Think: There are two triangles that meet the stated conditions. We will solve them separately. For each I can find ∠ A using an area formula. I then know two sides and the included angle, so I can then find side $a$ using the cosine rule.




$$Area = \frac{1}{2} bc \sin A$$

$$\therefore \quad \sin A = \frac{2 \times Area}{bc} = \frac{2 \times 85}{11 \times 18} = 0.8586$$

$$\therefore \quad A = \sin^{-1} 0.8586$$

$$A = 59.16^0 \quad \text{or} \quad 120.84^0$$

Now we solve both triangles.

$$a^2 = b^2 + c^2 - 2bc \cos A = 11^2 + 18^2 - 2 \times 11 \times 18 \times \cos 59.16^0 = 241.99$$

$$\therefore \quad a = \sqrt{241.99}$$

$$a = 15.6$$

The length of the 3rd side of the triangle is 15.6 cm.

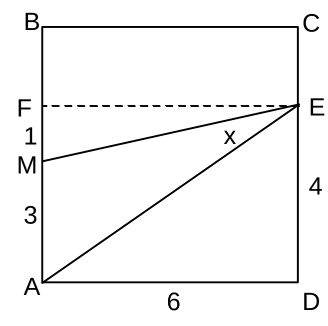
$$a^2 = b^2 + c^2 - 2bc \cos A = 11^2 + 18^2 - 2 \times 11 \times 18 \times \cos 120.84^0 = 648.01$$

$$\therefore \quad a = \sqrt{648.01}$$

$$a = 25.46$$

The length of the 3rd side of the triangle is 25.5 cm.

2. Think: Let the side of the square be 6 units, so lengths AM, DE and EC are all whole numbers. Let ∠ AEM = $x$. Draw EF parallel to CB. I can find ∠FEM and ∠AED using trig ratios. Once I know those I can find the required angle.




step 1. Find ∠FEM.

$$\tan f = \frac{\text{opp}}{\text{adj}}$$

$$\tan FEM = \frac{1}{6}$$

$$\therefore \quad \measuredangle FEM = \tan^{-1} \frac{1}{6}$$

step 2. Find ∠AED.

$$\tan f = \frac{\text{opp}}{\text{adj}}$$

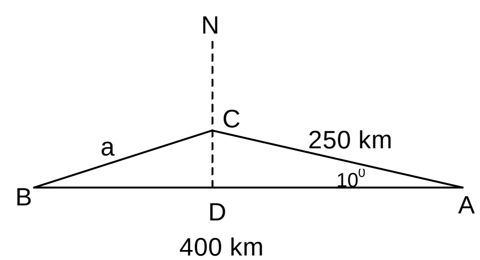
$$\tan AED = \frac{6}{4}$$

$$\therefore \quad \measuredangle AED = \tan^{-1}\frac{6}{4}$$

step 3. Find *x*.
Since ∠DEF is a right angle,

$$x = 90 - (\tan^{-1}\frac{1}{6} + \tan^{-1}\frac{6}{4})$$
$$= 24.2^0$$
$$= 24^0 13'40''$$

3. Think: In order to find the bearing from C to B we need to first find ∠ C. I know two sides and the included angle so I can use the cosine rule to find *a*. I can then use the sine rule to find C. Then I can find the bearing.




step 1. Find *a*.

$$a^2 = 400^2 + 250^2 - 2 \times 400 \times 250 \times \cos 10^0$$
$$a = 159.81$$

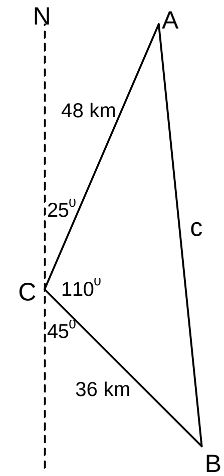
step 2. Find C.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$
$$\sin C = \frac{c \sin A}{a}$$
$$= \frac{400 \sin 10^0}{159.81}$$
$$C = \sin^{-1} 0.4346$$
$$C = 25.8^0 \quad \text{or} \quad 154.2^0$$

It is obvious from the diagram that C is $154.2^0$.

step 3. Find the bearing.
CB is on a bearing of 100°. Since BCD = 154.2°, the bearing of CD is 254.2°

4. Think: At 6 p.m., the first ship has travelled 48 km while the second ship has travelled 36 km. From the diagram, I need to find *c*. I know two sides and the included angle so I can use the cosine rule.

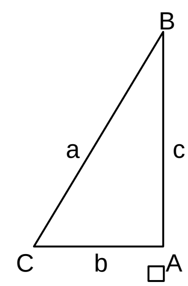



$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
&= 48^2 + 36^2 - 2\times 48\times 36\times \cos 110^0 \\
&= 4782 \\
\therefore\quad c &= \sqrt{4782} \\
&= 69.2
\end{aligned}
$$

Since the range of radio communication is 75 km, the ships will be able to communicate at 6 p.m.

5. Think: Comparing the two formulas, they both have a common factor of ½*b*. Therefore I need to show that $h = a\sin C$. I start by drawing diagrams.

a. In the diagram alongside, *h* = c.
But also




$$\sin C = \frac{opp}{hyp}$$

$$\sin C = \frac{c}{a}$$

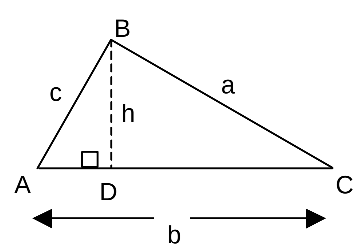
$\therefore \quad c = a\sin C$

But $\quad$ c = h

$\therefore \quad h = a\sin C$

$\therefore$ The two area formulas are equivalent for any right-angled triangle.

b. In the diagram alongside, BD is drawn perpendicular to AC. From triangle BCD,

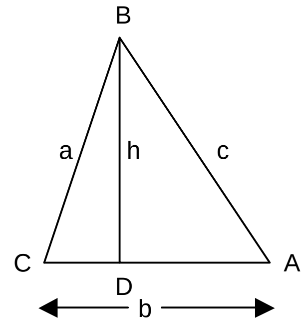



$$\sin C = \frac{opp}{hyp} = \frac{h}{a}$$

$\therefore \quad h = a\sin C$

$\therefore$ The two area formulas are equivalent for any triangle.

6. We shall prove instead $b = a\cos C + c\cos A$ (same thing, just different labels for the vertices and sides). In the diagram alongside the base CA consists of two parts, CD and DA. CD is the base of the right angled triangle with $\angle C$ and hypotenuse $a$. DA is the base of the right-angled triangle with $\angle A$ and hypotenuse $c$.




So $\quad \dfrac{DC}{a} = \cos C$

and so $\quad DC = a\cos C$

and $\quad \dfrac{DA}{c} = \cos A$

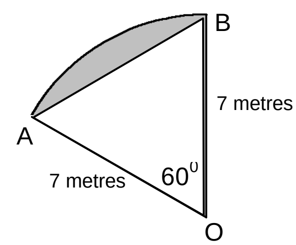
and so $\quad DA = c\cos A$

Now CD + DA = $b$, so we have

$$c\cos A + a\cos C = b$$

as required.

7. The circle can be thought of as six congruent sectors. One of the curved blue areas equals the area of a sector minus the area of the triangle formed by the two radii and the chord. The required area required is six times this.




Now the area of the circle is:

$$\begin{aligned} Area &= p r^2 \\ &= p(7)^2 \\ &= 49p \text{ sq metres.} \end{aligned}$$

We then calculate the area of each triangle:

$$\begin{aligned} Area &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}7^2\sin(60^\circ) \\ &= \frac{\sqrt{3}}{4}49 \end{aligned}$$

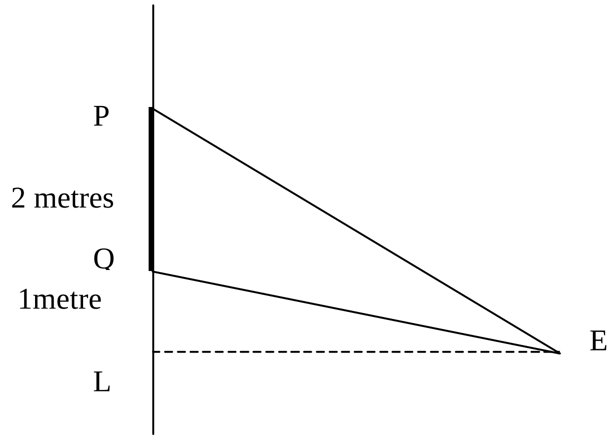
The hexagon consists of six triangles, so the area of the hexagon equals

$$\begin{aligned} Area &= 6\times\frac{\sqrt{3}}{4}49 \\ &= \frac{3\sqrt{3}}{2}49 \end{aligned}$$

Finally, calculate the area of the circle minus the area of the hexagon:

$$Area = 49p - \frac{3\sqrt{3}}{2}49 = 49(p - \frac{3\sqrt{3}}{2}) \simeq 26.63 \text{ sq.metres}$$

8.




There are two solutions, though only 1 seems feasible for a person with normal eyesight.

Let angle LEQ equal $\theta$ If the angle of viewing is 20° = $\pi/9$, then PEL = $\theta + \pi/9$.

Since QL = 1 metre, applying the definition of the tangent of an angle to triangle QLE , we have
QL/LE = tan($\theta$) , so
LE = QL/tan($\theta$) = 1/tan($\theta$).

Now the same argument applied to triangle PLE gives

LE = PL/tan($\theta + \pi/9$) = 3/tan($\theta + \pi/9$).

Thus the angle $\theta$ (i.e. QEL) satisfies the equation

$$\frac{1}{\tan\theta} = \frac{3}{\tan(\theta + \pi/9)}$$

We cannot solve this equation analytically, so we will solve it graphically. The graph of

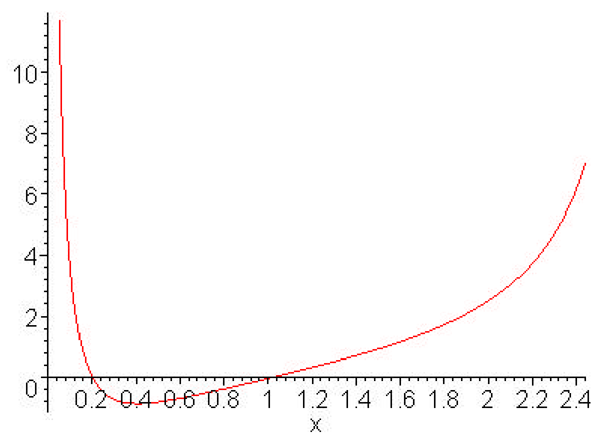
$$y = \frac{1}{\tan\theta} - \frac{3}{\tan(\theta + \pi/9)}$$

appears below.

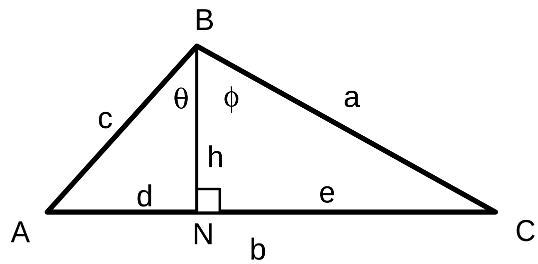
The graph cuts the $x$ axis at $x$ = 0.2 and 1.

For $x$ = 0.2, LE is approximately 4.9 metres, while for $x$ = 1, LE is approximately .64 metres.

The latter is surely too close, so we conclude that the person should stand about 5 metres from the picture.



9.




a. Now $\sin q = \frac{d}{c}$

so $d = c\sin q$

Similarly $\sin f = \frac{e}{a}$

so $e = a\sin f$

Now we will use $\text{Area} = \frac{1}{2}ab\sin C$ to find the area of the three triangles.

$$\text{Area}_{\text{ABC}} = \frac{1}{2}ac\sin(q+f)$$

$$\text{Area}_{\text{ABN}} = \frac{1}{2}dh = \frac{1}{2}hc\sin q$$

$$\text{Area}_{\text{CBN}} = \frac{1}{2}eh = \frac{1}{2}ha\sin f$$

We see that $\text{Area}_{ABC} = \text{Area}_{ABN} + \text{Area}_{CBN}$

so $\frac{1}{2}ac\sin(q+f) = \frac{1}{2}hc\sin q + \frac{1}{2}ha\sin f$ {substitute}

$ac\sin(q+f) = hc\sin q + ha\sin f$ {multiply through by 2}

$\frac{ac\sin(q+f)}{ac} = \frac{hc\sin q}{ac} + \frac{ha\sin f}{ac}$ {divide through by *ac*}

$$\sin(q+f) = \frac{h}{a}\sin q + \frac{h}{c}\sin f$$

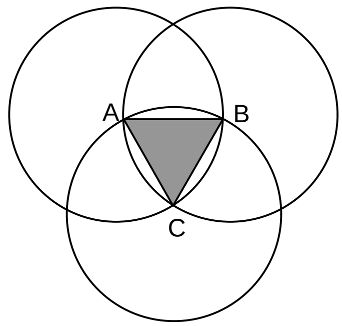
b. Now from the diagram

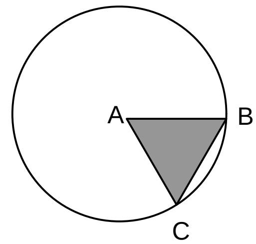
$\cos f = \frac{h}{a}$ and $\cos q = \frac{h}{c}$.

Substituting $\sin(q+f) = \cos f \sin q + \cos q \sin f$

Re-writing $\sin(q+f) = \sin q \cos f + \sin f \cos q$

10. Look at the diagram below, on the left. The shape marked by the points A, B and C consists of a central equilateral triangle and three congruent **segments**. We need to find all of these areas.




The lengths AB, BC, AC are all equal to the radii of the circle, *r*.

Since ABC is an equilateral triangle all its angles are 60°. Thus the area of the triangle (call the area *V*) is

$$V = \frac{1}{2} r^2 \sin 60^0 = \frac{1}{2} r^2 \frac{\sqrt{3}}{2} = r^2 \frac{\sqrt{3}}{4}$$

The area of each segment equals the area of the sector minus the area of the triangle. Since the sector is one-sixth of a circle of radius *r*, we can find an expression for area *G*:

$$G = \frac{1}{6} \times p r^2 = \frac{p}{6} r^2$$

so the area of the segment, *S,* is given by

$$\begin{aligned} S &= G - V \\ &= \frac{p}{6} r^2 - \frac{\sqrt{3}}{4} r^2 \\ &= (\frac{p}{6} - \frac{\sqrt{3}}{4}) r^2 \end{aligned}$$

The area we are to evaluate equals the area of the black triangle ABC plus 3 times the area of a sector, i.e. $V + 3S$

Thus

$$\begin{aligned} V + 3S &= r^2 \frac{\sqrt{3}}{4} + 3(r^2 (\frac{p}{6} - \frac{\sqrt{3}}{4})) \\ &= r^2 (\frac{\sqrt{3}}{4} + 3(\frac{p}{6} - \frac{\sqrt{3}}{4})) \\ &= r^2 (\frac{\sqrt{3}}{4} - 3\frac{\sqrt{3}}{4} + 3\frac{p}{6}) \\ &= r^2 (-2\frac{\sqrt{3}}{4} + \frac{p}{2}) \end{aligned}$$

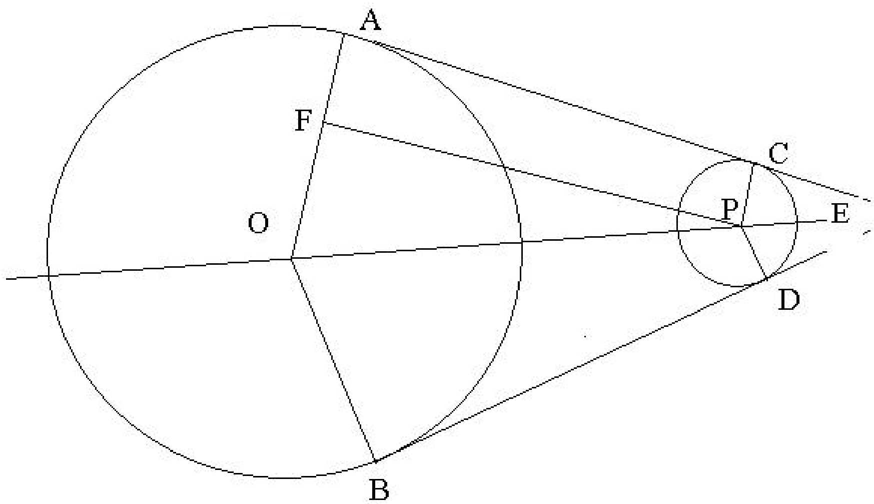
$$= r^2\left(\frac{\sqrt{3}}{4} + 3\left(\frac{p}{6} - \frac{\sqrt{3}}{4}\right)\right)$$

$$= r^2\left(\frac{p - \sqrt{3}}{2}\right)$$

The required area is given by:

$$\text{Area} = r^2\left(\frac{p - \sqrt{3}}{2}\right)$$

11. There is an error in the question in the book. The belt length in this problem is given by $(a + b)\pi + 2(a - b)\theta + 2c\cos\theta$ . Note the extra factor *q*.




In the diagram above, the length of OA = $a$, the length of OP = $c$, and the length of PC = $b$.

Because the belt is wrapped around each pulley and the lengths AC and BD are surely straight, each of AC and BD are tangents to the circles which represent the pulleys.

Because AC is a tangent to each circle, the radii OA and PC are perpendicular to AC. This means that OA and PC are parallel.

To find the length of the belt we need to find the lengths of the two straight line segments, AC and BD, and two circular arcs, specifically, the longer arc from A to B, and the shorter arc from C to D.

The key to finding these lengths is to draw a line from P to AO parallel to AC as shown in the diagram. Then, since the tangents and radii are perpendicular, the quadrilateral ACPF is a rectangle. Applying Pythagoras theorem to triangle OFP we can find the length of FP and therefore AC. When this is done we will know the lengths of all the straight line segments in the diagram, so we just need to find the lengths of the circular arcs.

To find the lengths of the circular arcs we need to know the angles subtended by the arcs AB and CD at the centre of the circle, i.e. we need to know the angles AOB and CPD. From the geometry, once we know one angle in the diagram we can find any of them.

If we let angle FPO be θ,

since $\quad \mathrm{OF} = a - \mathrm{b}$

we have $\quad \sin\theta = \dfrac{a-b}{c}$.

By symmetry angle CPE is half the angle CPD. Also, because FPC is a right angle, angles FPO and CPE sum to 90 °.

Thus $\quad$ angle CPE = $\pi/2 - \theta$

and $\quad$ angle CPD = $\pi - 2\theta$ .

Finally we need the reflex angle APB. The obtuse angle APB equals the obtuse angle CPD (these being corresponding angles of a transversal cutting parallel lines).

Thus the reflex angle

$$\mathrm{APB} = 2\pi - (\pi - 2\theta) = \pi + 2\theta .$$

Now to complete the problem,

belt length = 2( AC) + longer arc AB + shorter arc CD.

Now $\quad$ FP = AC

and from triangle FPO,

$$\mathrm{FP} = \sqrt{(a-b)^2 + c^2}$$

or $\quad \dfrac{\mathrm{FP}}{c} = \cos\theta$ ,

so $\quad \mathrm{FP} = c\cos\theta$

Since the arc of a circle equals the product of the angle at the centre and the radius, we have longer arc AB = $(\pi + 2\theta)a$ and shorter arc CD = $(\pi - 2\theta)b$ thus finally

$$\text{belt length} = (\pi + 2\theta)a + (\pi - 2\theta)b + 2c\cos\theta$$

and this can be rearranged to give

$$\text{belt length} = \pi(a+b) + 2\theta(a-b) + 2c\cos\theta$$

12. Looking at the diagram the distance $d$ is labelled, but it doesn't appear in the answer, so this suggests that the value of $d$ will be part of the solution.

Now we can relate B to the lengths $h$ and $d$ since the ground and the cliff are the sides in a right angled triangle   Similarly we can relate A, $d$ and $t+h$.

If the definition of the tangent of an angle is applied to B, we have

$$\tan B = \frac{h}{d}$$

Similarly applying the formula to A we have

$$\tan A = \frac{t+h}{d}$$

So $$d = \frac{h}{\tan B}$$

and substituting for $d$ in the second equation gives

$$\tan A = \frac{t+h}{\left(\frac{h}{\tan B}\right)}$$

We now simplify the above expression.

First be bring the term $\tan B$ into the numerator of the right side ('to divide by a fraction invert the fraction and multiply'). We have

$$\tan A = \frac{(t+h)\tan B}{h}$$

Now multiplying each side by $h$ gives

$$h\tan A = (t+h)\tan B$$

and dividing both sides by $\tan B$ gives

$$\frac{h\tan A}{\tan B} = (t+h)$$

and then $$\frac{h\tan A}{\tan B} - h = t$$

Finally factoring out $h$ gives the quoted formula.

$$t = h\left(\frac{\tan A}{\tan B} - 1\right)$$