Mathematics For Queensland

Year 11 Mathematics B, A Graphics Calculator Approach
Chapter Two – Quadratic Functions

Discovering the quadratic formula

Here are two approaches from master teachers:

I usually do an activity in which my Algebra I class tries to derive the formula. In fact, I plan on doing this after we get back from spring break.

First, the children are introduced and gain facility with using the "Completing the Square" strategy to solve quadratic equations.

After familiarity with completing the square is attained, a 2 – 3 day lesson with the goal of deriving the quadratic formula and then being able to use it in solving quadratic equations is implemented.

Day 1:

Using the completing the square strategy.

1. Outline and list the steps used in solving: $x^2 + 14x + 27 = 0$
2. Using the same steps as in problem 1, Solve the following quadratic for x: $x^2 + bx + c = 0$

We then "play" with the formula generated by solving some equations using the "formula" and/or completing the square.

Day 2:

1. Solve the equation $5x^2 + 11x + 2 = 0$
2. Solve the equation $ax^2 + bx + c = 0$

While most of the class struggles quite a bit on this day's activity, there are always a handful who can walk their way through the process and come up with a form of the equation. After the people who have "gotten it" have finished and moved on, and the frustration level of the rest of the class has been attained, the class is then "walked" through the derivation by the children who understand it as a class activity.

I have found this to be a very challenging activity for my advanced eighth graders and my ninth grade algebra students. However, the sense of accomplishment they show in being able to derive the simpler version (Day 1) and then struggling with the more general version makes the effort worthwhile for us all.

Not entirely on their own, of course, since it took the Western world several thousand years to get it perfectly straight. But a guided tour will lead to "discovery" of a sort when they get to the last step.

1. Have them solve $x^2 = 5$, etc. Also $x^2 = -5$, etc. You could say the latter doesn't have real solutions, or you could say whenever what occurs below the square root is a negative number, the solution is a complex number, and stop there. Later on, in the final step, the conditions on *a*, *b* and *c* which forbid real solutions will become obvious.

2. Have them solve $(x-3)^2 = 5$, etc.

3. Rewrite the above as $x^2 + 6x + 9 = 5$, noticing how lucky they were to have the left side a perfect square. Do a number of these. They can make them up, and then factor to make sure what they have made up is indeed a perfect square. It is also valuable to have them substitute each "solution" into the original equation to make sure it *is* a solution.

4. Rewrite the above as $x^2 + 6x + 4 = 0$, noticing that this is a pretty lucky arrangement even if they wouldn't think so at first. How could they have figured out how to convert this to the equation in (3), consequently the one in (2), which isn't much different from the one in (1) really. Work out some examples. Try $x^2 + 5x + 4 = 0$, noticing that you can still do it, but the result isn't very neat. In this case it is $x^2 + 5x + \frac{25}{4} = ?$. Spend a little time factoring "perfect squares" whose coefficients are not necessarily integers. For these messier equations it is even more valuable to go through the algebra of substituting each "solution" into the equation to make sure it is satisfied. The algebraic exercise alone, especially with fractions, is worthwhile.

5. Suppose it were $3x^2 + 15x - 12 = 0$; isn't it wise to divide through by 3 to begin with, to get one of the types being considered above?

6. Suppose it were $3x^2 - 4x + 5 = 0$; things get messier, but not impossible. Do a lot of these.

7. Ask: do you see the system? Make sure they do. Could they do it with *any* three coefficients, even if it leads to nasty fractions and all that? Let a student make up his own examples out of thin air, and don't let him tell you he can't take half of 5/3 (or whatever); he can. He can even square the result. If he can't, he is not ready for the next step.

8. How about $ax^2 + bx + c = 0$? If they can do it with *any* coefficients, here's their chance. The letters *a*, *b*, and *c* must not be mentioned before the students understand steps 1-7 thoroughly with strictly numerical coefficient examples. "Discovering" the quadratic formula is really a ratification of something already understood through numerical examples, and will be impossible for any student who doesn't understand what was being done in *all* the earlier numerical examples.

This is an outline; the lessons represented by all this should take a couple of weeks, I should think, depending on their previous experience with algebraic manipulations in general. It seems like a lot of time to spend on a single topic (solving quadratic equations), but the time is really being spent on much more, for the algebraic lessons in this sequence of generalizations are useful throughout all later mathematics, even the messier operations of handling fractions, taking differences of fractions using a common denominator, and so on.

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