Mathematics for Queensland, Year 12
Worked Solutions to Exercise 10H

Hints

1. A $500 excess means that the person insured pays the first $500 of each claim.

2. Sometimes it is easier to solve expected value questions if we consider the typical outcome of N trials. The value of N is chosen to avoid nasty fractions (some would say that this was a oxymoron. What is an oxymoron, you ask? Look it up!) For example, when throwing 2 dice, we often consider the typical outcome of 36 trials, since there are 36 equally likely outcomes.

   Try to answer this question by considering the typical outcome of 64 throws.

3. We need to determine the probability of getting 140 heads in 250 spins of a coin, assuming that the coin is not biased.

4. From the normal distribution tables or from your graphics calculator, the z-score that has 15% of the scores above it is $z = 1.0364$.

5. What is the probability of a value being more than 6σ from the mean?

6. Consider the area under the curve $y = f(x)$ between a and b. Call this area N. Then the area under the curve $y = 2f(x)$ is $2N$, and in general the area under the curve $y = a\ f(x)$ is aN. If you think of the area as being composed of thin rectangles, the reason for this should be clear.

   What is the area under $y = e^{-0.1x}$?

7. By symmetry, stanine 5 is centred on z = 0. If stanine 5 has a width of 0.5 standard deviations, what two z-scores is it between?

8. If we assume that the hand used makes no difference to the time, then we can treat this as a binomial distribution with $n = 20$ and $p = ½$.

9. Assume a fair coin ($p = 0.5$) is tossed 20 times. How many heads are needed to reject the claim that the coin is fair?

   Now assume that the coin is biased ($p = 0.5$). What is the chance of getting that many heads or more?

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1. Since there is a $500 excess, the average cost of a repair to the insurance company is $2400 - $500 = $1900.

   $E(x) = 350 - 0.08 \times 1900 - 0.01 \times 3800 = 160$.

   The insurance company can expect on average to earn $160 on each comprehensive cover premium.

2. Since P(Heads winning) = P(Tails winning), the expected return is the same whether a player bets Heads or bets Tails.

   Now the P(Odds occurring 5 times in a row) $= (\frac{1}{2})^5 = \frac{1}{32}$.

   Consider the typical outcome of playing 64 games, where the player bets $1 in each game.

   Odds will occur in 2 games, and the player loses $1 each time.

   Of the remaining 62 games, the player will win half, or 31 times. Each time he earns $1, for a total gain of $31.

   The player will also lose 31 times. Each time he loses $1, for a total loss of $31.

   The expected value $E(x)$ after 64 games is $E(x) = 31 - 31 - 2 = -2$.

   That is, for every $64 bet, on average the loss to the player is $2.

   Hence, the expected loss per $1 is $\frac{2}{64}$, or 0.03125, or a bit over 3 cents in the dollar.

   Therefore the expected return on a $1 bet is a bit less than 97 cents.

   N.B. The expected value equals Pr(Odds occurring 5 times in a row) $= (\frac{1}{2})^5 = \frac{1}{32} = 0.03125$. This is not a coincidence. Pr(Odds occurring 5 times in a row) is called the "house advantage" and is the profit that the casino earns on this game.

3. We need to determine the probability of getting 140 heads in 250 spins of a coin, assuming that the coin is not biased.

   $$\begin{aligned}\Pr(140\ heads) &= {}^{250}C_{140} \times 0.5^{140} \times 0.5^{110}\\ &= 0.0083\end{aligned}$$

   The probability of getting 140 heads in 250 spins of an unbiased coin is 0.0083. This is highly unlikely, so we conclude that the coin is biased towards heads.

4. From the normal distribution tables or from your graphics calculator, the z-score that has 15% of the scores above it is $z = 1.0364$.

   $$\begin{aligned}z &= \frac{x-m}{s}\\ 1.0364 &= \frac{x-300}{60}\\ x-300 &= 1.0364\times 60\\ x-300 &= 62.184\\ x &= 362.184\end{aligned}$$

The minimum mark should be 362 if the examiners are willing to accept slightly more than 15% of the applicants, or the minimum mark should be 363 if no more than 15% are accepted.

5. a. To a statistician, the term 'Six Sigma Quality' implies that the probability of an error (in a company's manufacturing process, or their service) should equal the area under the normal curve that is less than 6σ below the mean or more than 6σ above the mean. The tables in the textbook do not cater for such an extremely small probability, so we must use a graphics calculator.

    2*normalcdf(-1E99, -6) = 1.98024E-9 = 0.00000000198

    I would expect about 2 items in 1 billion to be faulty.

   b. "3.4 defects per million opportunities" is 1700 times as big as 2 per billion. It doesn't appear that this website has obtained Six Sigma Quality.

6. We will solve this using a useful property of area under a curve. Consider the area under the curve $y = f(x)$ between a and b. Call this area N. Then the area under the curve $y = 2f(x)$ is $2N$, and in general the area under the curve $y = a\ f(x)$ is aN. If you think of the area as being composed of thin rectangles, the reason for this should be clear.

   Now from a graphics calculator, fnInt(*e*^(-0.1*x*), *x*, 0, 10) = 6.3212. Hence the area under the curve $y = e^{-0.1x}$ is 6.3212. This area is 6.3212 times that required. Therefore the value of A = 1/6.3212 = 0.1582.

   As a check, fnInt(0.1582**e*^(-0.1*x*), *x*, 0, 10) = 1

7. To have a width of 0.5 standard deviations, stanine 5 is between $z = -0.25$ and $z = 0.25$. Then stanine 6 is between $z = 0.25$ and $z = 0.75$, and so on. We can use the symmetry of the normal distribution to reduce the number of calculations required.

Stanine	Calculation	Proportion
1	normalcdf(-1E99, -1.75)	0.040
2	normalcdf(-1.75, -1.25)	0.066
3	normalcdf(-1.25, -0.75)	0.121
4	normalcdf(-0.75, -0.25)	0.175
5	normalcdf(-0.25, 0.25)	0.197
6	normalcdf(0.25, 0.75)	0.175
7	normalcdf(0.75, 1.25)	0.121
8	normalcdf(1.25, 1.75)	0.066
9	normalcdf(1.75, 1E99)	0.040

8. If we assume that the hand used makes no difference to the time, then we can treat this as a binomial distribution with $n = 20$ and $p = ½$. Now 15 people were faster with their right hand than their left, so from the cumulative binomial distribution table,

   P(15 or more successes in 20 trials) = 1 – 0.9793 = 0.0207.

   The probability of this occurring by chance alone is only 0.0207, so we conclude that right-handed people generally can unscrew a lid with their right-hand faster than with their left.

9. If a fair coin ($p = 0.5$) is tossed 20 times, the probability of getting 13 or more heads is 0.0577. The probability of getting 14 or more heads is 0.021, so (using an alpha level of 0.05) we would claim that the coin is biased if it comes up head 14 times or more.

   The question now is, “What is the probability that the biased coin will come up heads 14 times or more?” The probability of 14 or more heads when $p = 0.6$ is 0.250.

   Conclusion: Using an alpha level of 0.05, the coin would be detected as being biased only 25% of the time.