Mathematics for Queensland, Year 12
Worked Solutions to Exercise 2J

Hints

1. Integrate $4 - 2x$ and evaluate it between 0 and a. Solve the resulting equation for a.

2. The only numerical technique that Maths B students have been taught is the trapezoidal rule. While not permitted to use the numerical integration function of your graphics calculator, you will need to use the calculator to evaluate the function at each of the ordinate lines.

3. $f(0) = 3$ gives you information about the y-intercept and hence the value of c.
Since you are also given information about the derivative and the definite integral, it may be necessary to find both of those for the given function.

4. To find the original function, you will need to differentiate twice. Don't forget the constant of integration each time!

5. To find the displacement, given the acceleration, you will need to differentiate twice. Don't forget the constant of integration each time!

6. The graph of $f(x) = -x^2 + h$ is symmetric about the y-axis. Draw a sketch so you can see what is going on. You will need to know the roots of the function in order to evaluate the definite integral.

7. Generalise your working in question 6 by letting the area be n.

8. This is a VHA+ question, as the algebra is quite extensive. Persistence will bring its own rewards. Along the way you may need to use the identity called the 'difference of cubes':
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

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1. $\int_0^a (4-2x)\ dx = 4$

$\left[4x - x^2\right]_0^a = 4$	{integrate}
$4a - a^2 = 4$	{substitute}
$a^2 - 4a + 4 = 0$	{collect terms on LHS}
$(a-2)^2 = 0$	{factorise}
$a = 2$	{solve for a}

2. We will use the Trapezoidal Rule, with 10 "strips"

x	0	π/10	2π/10	3π/10	4π/10	5π/10	6π/10	7π/10	8π/10	9π/10	π
y	0	.3090	.5878	.8090	.9511	1	.9511	.8090	.5878	.3090	0

Due to the symmetry of the graph, we can find the area between 0 and 5π/10 and double our result.

$$\int_0^{p} \sin x\ dx \simeq \frac{h}{2}[E + 2M]$$

Now $h = \pi/10$
$E = 1$
$M = 0.3090 + 0.5878 + 0.8090 + 0.9511 = 2.6569$

So $\int_0^{p} \sin x\ dx = \frac{p}{20}[1 + 2\times 2.6569]$
$= 0.9918$

Hence the total area under the sine curve from 0 to π is approximately 2 units^2.

3. If f(0) = 3, then

$3 = a(0^2) + b(0) + c$	{substituting}
$\therefore\ c = 3$	
Now $f'(x) = 2ax + b$	{find derivative}
$-3 = 2a(-1) + b$	{substitute}
$\therefore\ b = 2a - 3$	{Equation **1**}
Also $\int_0^1 ax^2 + bx + c\ \ dx = \frac{25}{6}$	{given}
so $\left[\frac{ax^3}{3} + \frac{bx^2}{2} + 3x\right]_0^1 = \frac{25}{6}$	{find definite integral}
$\frac{a}{3} + \frac{b}{2} + 3 = \frac{25}{6}$	{substitute}
$\frac{2a+3b}{6} = \frac{25}{6} - \frac{18}{6}$	{bring 3 to other side; get a common denominator}

$\frac{2a+3b}{6}=\frac{7}{6}$ {add fractions}

$2a+3b=7$ {Equation **2**}

Substituting **1** into **2**:

$2a+3(2a-3)=7$ {substituting}

$8a-9=7$ {expanding and collecting like terms}

$a=2$

Substituting into **1**:

$b=1$ {substituting}

Check using a graphics calculator.

4. $\frac{d^2y}{dx^2}=6x-2$

so $\frac{dy}{dx}=3x^2-2x+c$ {integrate}

Since gradient = 6 when x = 2:

$6=3(2)^2-2(2)+c$ {substituting}

$c=-2$ {solve for c}

$\therefore$ $\frac{dy}{dx}=3x^2-2x-2$

$\therefore$ $y=x^3-x^2-2x+d$ {integrate again}

Since y = 4 when x = 2:

$4=(2)^3-(2)^2-2(2)+d$ {substitute}

$\therefore$ $d=4$ {solve for d}

$\therefore$ $y=x^3-x^2-2x+4$

Check by working backwards.

5. $a=12t$

$\therefore$ $v=6t^2+c$ {integrate to get equation **1**}

$\therefore$ $s=2t^3+cx+d$ {integrate again to get equation **2**}

Substituting into **1**

$4=2(0)^3+c(0)+d$ {substituting}

$\therefore$ $d=4$ {solve for d}

And $0=2(1)^3+c(1)+d$ {substituting}

$\therefore$ $c+d=0$ {simplifying}

so $c=-4$ {substitute 4 for d and solve for c}

Hence the displacement s at time t is given by:

$s=2t^3-4x+4$

6. The graph of $f(x) = -x^2 + h$ is symmetric about the y-axis.

Find the roots of the function:

$-x^2 + h = 0$ {set $f(x) = 0$}

$\therefore\ x = \pm\sqrt{h}$ {solve for x}

So the area under $f(x) = -x^2 + h$ between $-\sqrt{h}$ and $\sqrt{h}$ equals 2.

By symmetry, the area between 0 and $x = \sqrt{h}$ between the function and the x-axis equals 1.

i.e. $\left[\dfrac{-x^3}{3} + hx\right]_0^{\sqrt{h}} = 1$ {integrate}

$\dfrac{-\left(\sqrt{h}\right)^3}{3} + h\left(\sqrt{h}\right) = 1$ {substitute}

$\dfrac{-\left(\sqrt{h}\right)^3}{3} + \dfrac{3h\left(\sqrt{h}\right)}{3} = 1$ {get a common denominator}

$2h^{\frac{3}{2}} = 3$ {multiply by 3; write in index notation}

$h^{\frac{3}{2}} = \dfrac{3}{2}$ {divide by 2}

$h = \left(\dfrac{3}{2}\right)^{\frac{2}{3}}$ {raise both sides to the 2/3 power}

$h \doteq 1.310$ {evaluate}

Check with a graphics calculator.

7. Since the area between $-\sqrt{h}$ and $\sqrt{h}$ equals n units^2, the area between 0 and $\sqrt{h}$ equals ½n

i.e.

$\left[\dfrac{-x^3}{3} + hx\right]_0^{\sqrt{h}} = \dfrac{n}{2}$ {from the working above}

Following the above working leads to:

$h^{\frac{3}{2}} = \dfrac{3n}{2}$ {from the working above}

So $h = \left(\dfrac{3n}{2}\right)^{\frac{2}{3}}$ {raise each side to the 2/3 power}

or $h = \left(\dfrac{3}{2}\right)^{\frac{2}{3}}(n)^{\frac{2}{3}}$ {index law 4}

so $h \doteq 1.310\, n^{\frac{2}{3}}$ {evaluating $(3/2)^{2/3}$}

8. a. $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} x^2 - \frac{1}{12} dx = \left[\frac{x^3}{3} - \frac{1}{12} x \right]_{r-\frac{1}{2}}^{r+\frac{1}{2}}$ {integrate}

$= \frac{(r+\frac{1}{2})^3}{3} - \frac{(r+\frac{1}{2})}{12} - \left(\frac{(r-\frac{1}{2})^3}{3} - \frac{(r-\frac{1}{2})}{12} \right)$ {substitute}

$= \frac{1}{3}(r+\frac{1}{2})^3 - \frac{1}{3}(r-\frac{1}{2})^3 - \frac{1}{12}(r+\frac{1}{2}) + \frac{1}{12}(r-\frac{1}{2})$ {re-arrange terms}

$= \frac{1}{3}\left[(r+\frac{1}{2})^3 - (r-\frac{1}{2})^3 \right] - \frac{1}{12}\left[(r+\frac{1}{2}) - (r-\frac{1}{2}) \right]$ {take out common factors}

$= \frac{1}{3}\left[(r+\frac{1}{2})^3 - (r-\frac{1}{2})^3 \right] - \frac{1}{12}$ {simplify 2nd term}

Now expand, using the difference of cubes:

$= \frac{1}{3}\left[(r+\frac{1}{2}) - (r-\frac{1}{2}) \right]\left[(r+\frac{1}{2})^2 + (r+\frac{1}{2})(r-\frac{1}{2}) + (r-\frac{1}{2})^2 \right] - \frac{1}{12}$

$= \frac{1}{3}[1]\left[(r^2 + r + \frac{1}{4}) + (r^2 - \frac{1}{4}) + (r^2 - r + \frac{1}{4}) \right] - \frac{1}{12}$ {simplify 1st factor;
expand 2nd factor}

$= \frac{1}{3}\left(3r^2 + \frac{1}{4}\right) - \frac{1}{12}$ {combine like terms}

$= \left(r^2 + \frac{1}{12}\right) - \frac{1}{12}$ {expand}

$= r^2$ {Q.E.D.!}

b. From the above derivation, the area from $x = \frac{1}{2}$ to $x = 1 + \frac{1}{2}$ is given by $\ ^2$.
the $x = 1 + \frac{1}{2}$ to $x = 2 + \frac{1}{2}$ is given by 2^2.
the area from $x = 2 + \frac{1}{2}$ to $x = 3 + \frac{1}{2}$ is given by 3^2.
etc.

Hence the total area from $x = \frac{1}{2}$ to $x = n + \frac{1}{2}$ is given by $1^2 + 2^2 + ... + n^2$.

Now we show that this area is also given by $\frac{n(n+1)(2n+1)}{6}$.

A $= \int_{\frac{1}{2}}^{n+\frac{1}{2}} x^2 - \frac{1}{12} dx = \left[\frac{x^3}{3} - \frac{1}{12} x \right]_{\frac{1}{2}}^{n+\frac{1}{2}}$ {integrate}

$= \left[\frac{(n+\frac{1}{2})^3}{3} - \frac{n+\frac{1}{2}}{12} \right] - \left[\frac{(\frac{1}{2})^3}{3} - \frac{1}{12} \times \frac{1}{2} \right]$ {evaluate}

$= \left[\frac{(n+\frac{1}{2})^3}{3} - \frac{n+\frac{1}{2}}{12} \right] - \left(\frac{1}{24} - \frac{1}{24} \right)$ {2nd term = 0}

$= \left[\frac{4(n+\frac{1}{2})^3 - (n+\frac{1}{2})}{12} \right]$ {common denominator}

$= \frac{(n+\frac{1}{2})\left[4(n+\frac{1}{2})^2 - 1 \right]}{12}$ {take out a common factor}

$= \frac{(n+\frac{1}{2})\left[4(n^2 + n + \frac{1}{4}) - 1 \right]}{12}$ {expand}

$$= \frac{(n+\tfrac{1}{2})\left[(4n^2+4n+1)-1\right]}{12} \qquad \{\text{expand}\}$$

$$= \frac{(n+\tfrac{1}{2})\left(4n^2+4n\right)}{12} \qquad \{\text{combine like terms}\}$$

$$= \frac{4(n+\tfrac{1}{2})n\left(n+1\right)}{12} \qquad \{\text{take out a common factor}\}$$

$$= \frac{(n+\tfrac{1}{2})n\left(n+1\right)}{3} \qquad \{\text{simplify by dividing}\}$$

$$= \frac{(\frac{n+1}{2})n\left(n+1\right)}{3} \qquad \{\text{common denominator}\}$$

$$= \frac{\frac{1}{2}(n+1)n\left(n+1\right)}{3} \qquad \{\text{take the ½ out front}\}$$

$$= \frac{n\left(n+1\right)(2n+1)}{6} \qquad \{\text{simplify and re-arrange}\}$$

Q.E.D.