Mathematics for Queensland, Year 12
Worked Solutions to Exercise 3P

Hints

1. Square both sides. Remember that $\sin^2 x + \cos^2 x = 1$

2. Expand the LHS, then write the expression in terms of sin *x* and cos *x*.
   Recall the identity for the perfect square: $(a \pm b)^2 = a^2 \pm 2ab + b^2$

3. Oops, this question is similar to the previous question.

4. ERRATA! The coordinates of the point P are $\left(\frac{5p}{6}, -\frac{\sqrt{3}}{2}\right)$
   To find the equation of a line, it is sufficient to know the gradient and the coordinates of one point.

5. To solve, write the equation in terms of either sin *x* only or cos *x* only.

6. Since the phase shift is 0 and the vertical shift is 0, the "waist equation" has the form $v = A\sin Bt$.

7. You know the displacement function, how do you find the velocity function. Note that the LHS is v^2 and not just *v*.

8. Oops! The question should read, "What is $\int f(x)\ dx$?
   But the question is also misplaced – it should also be at the end of Chapter 8.

   Here is an alternative question, along the same lines:

   If $y = x\sin x$, find $\frac{dy}{dx}$.

   Hence evaluate $\int_0^{\frac{p}{2}} x\cos x\ dx$

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1.

$$\sin A + \cos A = 1.2$$

$$(\sin A + \cos A)^2 = 1.2^2 \qquad \text{\{square both sides\}}$$

$$\sin^2 A + 2\sin A\cos A + \cos^2 A = 1.44 \qquad \text{\{expand\}}$$

$$2\sin A\cos A = 0.44 \qquad \text{\{since } \sin^2 A + \cos^2 A = 1\text{\}}$$

$$\sin A\cos A = 0.22$$

2.

$$(\sec A - \tan A)^2 = \sec^2 A - 2\sec A\tan A + \tan^2 A \qquad \text{\{expand\}}$$

$$= \frac{1}{\cos^2 A} - 2\frac{1}{\cos A}\frac{\sin}{\cos A} + \frac{\sin^2 A}{\cos^2 A} \qquad \text{\{write using } \sin A \text{ and } \cos A\text{\}}$$

$$= \frac{\sin^2 A - 2\sin A + 1}{\cos^2 A} \qquad \text{\{add fractions\}}$$

$$= \frac{(\sin A - 1)^2}{1 - \sin^2 A} \qquad \text{\{numerator is a perfect square\}}$$

$$= \frac{(\sin A - 1)(\sin A - 1)}{(1 + \sin A)(1 - \sin A)} \qquad \text{\{difference of squares\}}$$

$$= \frac{-(1 - \sin A)(\sin A - 1)}{(1 + \sin A)(1 - \sin A)} \qquad \text{\{ } \sin A - 1 = -(1 - \sin A) \text{ \}}$$

$$= \frac{-(\sin A - 1)}{(1 + \sin A)} \qquad \text{\{divide out } (1 - \sin A)\text{ \}}$$

$$= \frac{1 - \sin A}{1 + \sin A} \qquad \text{\{ Q.E.D.\}}$$

3. see above.

4. ERRATA! The coordinates of the point P are $\left(\frac{5p}{6}, -\frac{\sqrt{3}}{2}\right)$

To find the equation of a line, it is sufficient to know the gradient and the coordinates of one point.
To find the gradient:

$$m = \frac{dy}{dx} = 2\cos 2x$$

At the point $P(\frac{5p}{6}, 1)$

$$m = 2\cos\frac{2\times 5p}{6} = 2\cos\frac{5p}{3} = 2\times\frac{1}{2} = 1$$

The equation is

$$y - y_1 = m(x - x_1)$$

$$y + \tfrac{\sqrt{3}}{2} = 1(x - \tfrac{5p}{6})$$

$$y = x - \tfrac{5p}{6} - \tfrac{\sqrt{3}}{2}$$

$$y \doteq x - 3.484$$

Check this answer with a calculator.

5. Given $2\sin^2 x = \cos x + 1$

Write the equation in terms of cos *x* {since $\sin^2 x + \cos^2 x = 1$}

$2(1-\cos^2 x) = \cos x + 1$ {substitute}

$2 - 2\cos^2 x = \cos x + 1$ {expand}

$2\cos^2 x + \cos x - 1 = 0$ {re-arrange}

This is a quadratic equation on cos *x*.
Let $y = \cos x$

$2y^2 + y - 1 = 0$ {substitute *y* for cos *x*}

$(2y-1)(y+1) = 0$ {factorise}

$2y - 1 = 0$ or $y + 1 = 0$

$y = \frac{1}{2}$ or $y = -1$ {solve the quadratic equation}

Hence $\cos x = \frac{1}{2}$ or $\cos x = -1$ {substitute cos *x* for *y*}

Now solve these equations for *x*.

If $\cos x = \frac{1}{2}$ then $x = \frac{\pi}{3} + 2k\pi$, $k \in J$ or $x = \frac{5\pi}{3} + 2k\pi$, $k \in J$

If $\cos x = -1$ then $x = \pi + 2k\pi$, $k \in J$

6. The "waist function" has the form $v = A\sin Bt$ where

A = 0.6 {*A* = the amplitude}

B = $\frac{2\pi}{2} = \pi$ {$B = \frac{2\pi}{\text{period}}$}

So $v = 0.6\sin \pi t$

Acceleration is the derivative of velocity, so

$a = 0.6\pi\cos \pi t$ {take the derivative of *v*}

At time $t = 0.5$

$a = 0.6\pi\cos \pi(0.5)$ {substitute and evaluate}

$= 0$

At time $t = 0.5$, the acceleration is 0.

7. Displacement is given by

$x = a\sin nt$ **1**

Velocity is the derivative of displacement, so

$v = \frac{dx}{dt} = an\cos nt$ {take the derivative}

and $v^2 = a^2n^2\cos^2 nt$ {square both sides}

But $\cos^2 nt = 1 - \sin^2 nt$ {the Pythagorean identity}

so $v^2 = a^2n^2(1 - \sin^2 nt)$ {substitute}

$v^2 = n^2(a^2 - a^2\sin^2 nt)$ {re-arrange – do you see why?}

$v^2 = n^2\left[a^2 - (a\sin nt)^2\right]$ {since $x^2y^2 = (xy)^2$}

But $x = a\sin nt$ {**1** above}

$\therefore$ $v^2 = n^2\left(a^2 - x^2\right)$ {substitute}

Q.E.D.

8. If $y = x\sin x$ then using the product rule

$$\frac{dy}{dx} = \sin x + x\cos x$$

This has an extra term, $\sin x$. We can eliminate it by adding a cos x term to the RHS. As a check:

If $y = x\sin x + \cos x$, then

$$\begin{aligned}\frac{dy}{dx} &= (\sin x + x\cos x) - \sin x \\ &= x\cos x\end{aligned}$$

Hence
$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\cos x \; dx &= \left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}} \\ &= \left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) - \left(0\sin 0 + \cos 0\right) \\ &= \tfrac{\pi}{2} - 1\end{aligned}$$

Check with a graphics calculator.