Mathematics for Queensland, Year 12
Worked Solutions to Exercise 4I

Hints

1. The first die can be any number. What is the chance that the 2nd die matches the first? The 3rd die? What is the chance that the 2nd AND the 3rd AND the 4th AND the 5th match the first die?

2. a. What is the expected value of the game?

   b. If they had continued until there was a winner, what is the probability that Kate would win?

3. Errata! The equation of a sphere of radius *r* is given by $x^2 + y^2 + z^2 = r^2$.
   Randomly choose the 3 coordinates of a point inside the rectangle. Is the point also inside the hemisphere?

4. a. Errata! The first die should be labeled **A**.
   Compare, for example, die B and die C. What numbers on die B will win?
   What is an efficient way of showing the outcome when 2 dice are thrown?

   b. Look at the outcomes when one die is thrown with another. For most pairs of dice, it is possible to figure out which die, if any, has an advantage.

5. Which of these questions uses the Multiplication Principle? Which uses the Addition Principle? For the Addition Principle, are the two events mutually exclusive?

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7. If a green face is uppermost, what are the possible outcomes? How many of these have a green face on the other side of the card?

8. This is a gold standard question – at the VHA+ level.

   What is the minimum number of serves Patrick needs now to win the game? If p = probability he wins the next point, what is the probability that he wins the match in the minimum number of serves (answer in terms of p).

   How else may Patrick win the match? What is the probability that he wins that way?

   All you need to answer is "Might this be a realistic claim?." Do you need to consider all possible cases to answer this question?

9. Some trial and error might be a good way to start. Why are some arrangements better than others?

10. It is easier to consider the question "What is the probability that no students share the same birthday?."

Worked Solutions to Exercise 4I

1. The first die can be any number, so the probability of success on the first die is 1. Each of the remaining 4 dice has to match that number, with a probability of $\frac{1}{}$

$$= \times - \times - \times - \times - = — = ——$$

2. a. The expected value E = ($1 x 1) + (-$1 x 1) = 0. Therefore the game is fair.
   b. The fairest way to divide the kitty is to base it on the probability that each would win if they had finished the game.
   If they had continued after the first round, Kate needed to win the next two rounds to win the game.
   ∴ Pr(Kate wins the next two rounds) = ½ x ½ = ¼
   Pr(Jenna wins the next two rounds) = 1 - ¼ = ¾
   ∴ The fairest way to divide the kitty is
   Kate: $0.50
   Jenna: $1.50

3. Errata! The equation of a sphere of radius r is given by $x^2 + y^2 + z^2 = r^2$.
   To estimate the volume of the hemisphere, we will randomly choose a point inside the rectangle as follows:
   Randomly choose an x-value from –1 to 1.
   Randomly choose a y-value from –1 to 1.
   Randomly choose a z-value from 0 to 1.
   If $x^2 + y^2 + z^2 \leq 1$ then the point also lies inside or on the hemisphere; otherwise it lies outside the hemisphere.

   Using a graphics calculator to carry out a simulation, one of the authors found that 102 of 200 randomly chosen points lay inside of the hemisphere.

   Now the volume of a 2 x 2 x 1 rectangle is 4 units^2. So the estimated volume of the hemisphere is

   $$\frac{102}{200} \times 4 = 2.04 \text{ units}^2.$$

   The actual volume is given by

   $$V = \frac{4}{3} p r^3 \div 2$$
   $$= \frac{4}{3} p (1^3) \div 2$$
   $$= 2.094 \text{ units}^2$$

   In this experiment, the Monte Carlo method had a percentage error of
   $\frac{2.094 - 2.04}{2.094} \times 100\% = 2.58\%$. As the Monte Carlo method is a random process, you probably will get a somewhat different result.

4. a. Errata! The first die should be labeled **A**.

If your opponent chooses die B, you should choose die A. Here is a grid showing the outcomes. A means die A wins and B means die B wins.

Die B \ die A	2	2	2	2	6	6
1	A	A	A	A	A	A
1	A	A	A	A	A	A
1	A	A	A	A	A	A
5	B	B	B	B	A	A
5	B	B	B	B	A	A
5	B	B	B	B	A	A

Die A beats die B $\frac{2}{3}$ of the time.

Also, die C beats die B only wins ½ of the time (when die B shows a 1) and die D will lose to die B more than half the time (die B wins every time with a 5 and occasionally with a 1).

b. With some analysis, the answer can be determined without drawing a grid.

Opponent chooses	**You choose**	**P(you win)**	**Justification**
A	C	$\frac{2}{3}$	C will beat A every time A throws a 2.
B	A	$\frac{2}{3}$	see above.
C	D	$\frac{2}{3}$	D will beat C every time D throws a 4.
D	B	$\frac{2}{3}$	On ½ of the throws, B will win with a 5. And B also wins with a 1 if D throws a 0. This happens $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$ of the time. And $\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$.

5. a. i. Water flows at O if **both** A **and** B leak. We apply the multiplication principle.

Pr(water flows at O) = Pr(water flows at A) x Pr(water flows at B)
= 0.05 x 0.08 = 0.004.

ii. Water flows if **either** A or B or **both** leak. We apply the addition principle.

Pr(water flowing at O) = Pr(flow at A) + Pr(flow at B) – Pr(flow at both)
= 0.05 + 0.08 – 0.004
= 0.126

b. Pr(flow at O) = Pr(flow at C) x Pr(flow at A or B or both)
0.02 = Pr(flow at C) x 0.126

∴ Pr(flow at C) = $\frac{0.02}{0.126} = 0.159$

6. a. The chain of components works if the1[st] component works and the 2[nd] component works and the 3[rd] component works and the 4[th] component works and the 5[th] component works.
Pr(chain of components works) = 0.9 x 0.9 x 0.9 x 0.9 x 0.9 = 0.9^5 = 0.59049.

b. This is an application of the Addition Principle.
Pr(chain of components works) = Pr(top chain works) + Pr(bottom chain works) – Pr(both work)
= 0.59049 + 0.59049 = (0.59049 x 0.59049)
= 0.8323

c. Pr(a given pair works) = Pr(top works) + P(bottom works) – Pr(both work)
= 0.9 + 0.9 – 0.92 = 0.99
Pr(chain of components works) = 0.99 x 0.99 x 0.99 x 0.99 x 0.99 = 0.99^5 = 0.95.

7. This is sort of a trick question. One may think that the answer is ½, reasoning as follows: If the visible face is green then the card is either green-green or green-white. Therefore the other face has a 50:50 chance of being green.

But this is not correct! Label the sides of the cards W1:W2, G1:G2, W3:G3. Assuming that you see a green face on the card, then the reduced sample space is G1, G2 or G3. In two out of three cases, the other face is Green. The probability that the other face is Green is $\frac{2}{3}$.

8. Note: This is a gold standard question! In order to attempt this question, we need to assume that winning the points are independent events, so the probability that Patrick wins the next point is the same. Call this value p.

In order for Patrick to win, one of the following must occur:

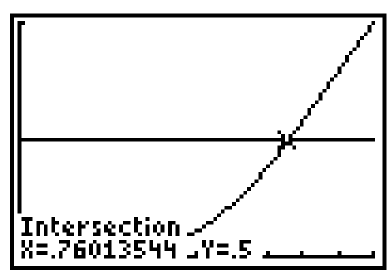
Outcome	**Probability**
He wins 4 straight points, which we write as WWWW	$P(x_1) = p \times p \times p \times p = p^4$
He wins 2 straight points (to get to deuce) and then wins 1 of the next 2 points (to get back to deuce) and then wins the next two points (to win the match). Note there are two ways to get back to deuce – WL and LW.	$P(x_2) = p \times p \times [2 \times p \times (1-p)] \times p \times p$ $= 2p^5(1-p)$
He wins 2 straight points (to get to deuce) and then wins 1 of the next 2 points (to get back to deuce), then again wins 1 of 2 points and finally wins the next two points (to win the match).	$P(x_3) = p \times p \times [2 \times p \times (1-p)] \times [2 \times p \times (1-p)] \times p \times p$ $= 4p^6(1-p)^2$
And so on.	

One small problem is there are an infinite number of ways that Patrick can win (he can win after 4 points are played, or 6 points, or 8 points, or ...). We will simplify the problem by only considering those cases listed above.

Since the events above are mutually exclusive, P(Patrick wins) = P(x_1) + P(x_2) + P(x_3)

Since Fred Stolle said Patrick has a 50% chance of winning, we can write,

$$P(x_1) + P(x_2) + P(x_3) = 0.5$$



This is best solved by graphics calculator. Let

$$Y1 = x^4 + 2x^5(1-x) + 4x^6(1-x)^2$$
$$Y2 = 0.5$$

and find the point where they intersect.

If Patrick can win about 76% of the points, he will have a 50% chance of winning the game. This is a realistic claim.

Note to Maths C students: In your course you will learn about a topic called the "sum of an infinite geometric series." You can use what you learn in this topic to show that the exact answer is p = 0.7467.

9. Some thought, and maybe some trial and error, is needed to arrive at the following maximal arrangement:
   In Box A, put 1 vial of water
   In Box B, put 1 vial of water
   In Box C, put the remaining 4 vials of water and 6 vials of poison.

   If the evil king choose Box A or Box B, the victim will survive. If the evil king chooses Box C, there is still a 0.4 chance of surviving.

   Pr(victim is poisoned) = Pr(Box C is chosen) x Pr(poison vial is chosen) = $\frac{1}{3}\times\frac{6}{10}=\frac{1}{5}$.

   Therefore, Pr(victim survives) = $1-\frac{1}{5}=\frac{4}{5}$.

   To minimise the chance of survival, just swap the poison vials and the water vials. Now Pr(victim survives) = $\frac{1}{5}$.

10. This is a classic probability question with a surprising answer. It is best tackled by asking (and answering) the complement of the given question, i.e. "What is the probability that no students share the same birthday?"

    Assume there are 365 days in a year (i.e. ignore leap year) and by share a birthday we mean they are born on the same date but not necessarily the same year.

    If no students share a birthday, then (2^{nd} student doesn't share a birthday with the first) AND (the 3^{rd} student doesn't share a birthday with the 1^{st} or 2^{nd}) AND (the 4^{th} student doesn't share a birthday with the 1^{st}, 2^{nd} or 3^{rd}), etc. This is an application of the multiplication principle, as follows:

    Pr(no students share a birthday) = Pr(2^{nd} student doesn't share) x Pr(3^{rd} student doesn't share) x Pr(4^{rd} student doesn't share) x P.r.(24^{th} student doesn't share)

    Pr(no students share a birthday) = $\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times\frac{361}{365}\times\ldots\times\frac{342}{365}\doteq 0.5$

    Therefore, Pr(at least 2 students share a birthday) = 1 – Pr(no students share a birthday)

    = 1 – 0.5 = 0.5

    Most people find this probability surprisingly high. It might be even more surprising to learn that in a group of 30 people, there is a 71% chance that two or more people share a birthday. And in a group of 60 people, the probability is greater than 99%!