Mathematics for Queensland, Year 12
Worked Solutions to Exercise 5L

Hints

1. There isn't an algebraic solution to this problem. It needs to be tackled using either graphs or a table. Your choice!

2. Find an expression for the value of the Mackay property, in terms of *t*. Also find an expression for the value of the Noosa property, in terms of *t*.

   You are interested in when these two function interest.

3. The answer is not 3 cm! With some thought, it should be clear that each centimeter of charcoal removes 33% of the remaining quantity of chemicals, not 33% of the original quantity of chemicals.

4. If there are N_0 germs initially, and ½ are killed by disinfectant, and then the number of germs remaining increases by 1/3, what is the expression for the number of germs at the start of day 1?

5. If you double the number of speakers, you double the intensity of the sound. Remember - a sound with an intensity of 110 dB has *10* times the intensity of a sound with an intensity of 100 dB.

6. Let the current oil reserves equal 100 units. If the analysts say that the oil will last for 100 years at the current rate of consumption, then they are assuming that we use 1 unit each year. But if consumption is increasing at 4% per year, then in the 1st year we use 1 unit of oil. But how much do we use in the next year? And the next?

   Use a spreadsheet to track the number of units of oil each year and the total amount of oil used, and to determine when all 100 units have been used.

7. Start by finding a mathematical model for the amount *A* of space junk being added each year *t* from 1991 to 2000. Taking 1990 to be year 0, the weight of space junk when $t = 0$ is 1.8 million kg. Hence the amount of space junk added each year is assumed to be an exponential function that passes through (1, 0.8) and (10, 1.2).

   Once you have found this function, use a spreadsheet to track the weight of space junk being added each year, and the total weight of space junk each year.

8. Here "solve for *n*" means to find an algebraic expression for *n* in terms of *a*, *b* and *m*.

   Use logs to "get *n* onto the ground". Bring all terms involving *n* to the same side of the equation.

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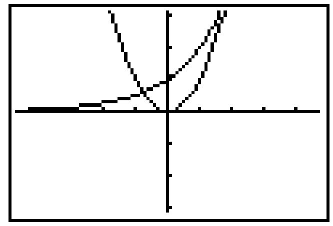
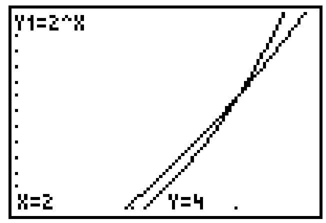
1. As there isn't an analytical (i.e. algebraic) solution to this equation, we will solve it by sketching graphs and finding the points of intersection.



Setting Y1 = e^x and Y2 = x^2 and selecting (Zoom, Decimal) on a TI-83 brings up the screenshot alongside. It shows 2 points of intersection. Using (Calc, Intersect) shows us that 2 solutions are:

$x \doteq -0.77, \;\; y \doteq 0.58$

$x = 2, \;\; y = 4$



By changing the window, we find a 3rd solution:

$x = 4$, $y = 16$

Have we found *all* of the solutions? We have, because every exponential function eventually grows faster than any power function. This means that $e^x > x^2$ for $x > 4$ and hence there are no more points of intersection.

2. Here is an algebraic solution. A graphical solution is equally valid.

Mackay: $A = 400\,000(1.04)^n$

Gold Coast: $A = 250\,000(1.06)^n$

$400\,000(1.04)^n = 250\,000(1.06)^n$	{Equate the 2 RHS expressions}
$40(1.04)^n = 25(1.06)^n$	{Divide by 10 000}
$1.6(1.04)^n = (1.06)^n$	{Divide by 25, and convert to a decimal}
$\log[1.6(1.04)^n] = \log[(1.06)^n]$	{Take log of both sides}
$\log 1.6 + n\log 1.04 = n\log 1.06$	{Apply log laws}
$\log 1.6 = n\log 1.06 - n\log 1.04$	{Bring terms with n together}
$n(\log 1.06 - \log 1.04) = \log 1.6$	{Factor out the common factor of n}
$n = \dfrac{\log 1.6}{\log 1.06 - \log 1.04}$	{Solve for n}
$n \doteq 24.7$ years	{Evaluate}

3. The answer is not 3 cm! With some thought, it should be clear that each centimeter of charcoal removes 33% of the remaining quantity of chemicals, not 33% of the original quantity of chemicals.

Let the original mass of chemicals = M_0.

After 1 cm:	$M = M_0 \times 0.67$	{If 33% is removed, 67% remains}
After 2 cm:	$M = M_0 \times 0.67 \times 0.67$	{Now 67% of 67% of M_0 remains}
After 3 cm:	$M = M_0 \times 0.67 \times 0.67 \times 0.67$	{Etc}
…		
After n cm:	$M = M_0 \times 0.67^n$	{The pattern should be clear}

When 99% is removed:

$0.01M_0 = M_0 \times 0.67^n$	{When 99% is removed, 1% remains}
$0.01 = 0.67^n$	{Divide both sides by M_0}
$\log 0.01 = \log 0.67^n$	{Take the log of both sides}
$\log 0.01 = n \log 0.67$	{Apply log law 3}
$n = \dfrac{\log 0.01}{\log 0.67}$	{Solve for n}
$n = 5.75\,\text{cm}$	{Evaluate}

About 6 cm of charcoal are needed to remove 99% of the chemicals.

4. Let N_0 be the number of germs in the room at time t = 0.

After the disinfectant is applied, the number of germs $N = \frac{1}{2}N_0$.

At the start of day t = 1, the number of germs has grown to

$$N = \frac{1}{2}N_o + \frac{1}{3}\left(\frac{1}{2}N_o\right) = \frac{1}{2}N_o + \frac{1}{6}N_o = \frac{2}{3}N_o.$$

Each day the number of germs is $\frac{2}{3}$ of the number of germs on the previous day.

Hence on day t,

$$N = \left(\frac{2}{3}\right)^t N_o$$

When N = 0.001 N_0,	{When only $\frac{1}{1000}$ of the germs remain}
$0.001N_0 = \left(\frac{2}{3}\right)^t N_o$	{Substitute}
$0.001 = \left(\frac{2}{3}\right)^t$	{Divide by N_0}
$t = \dfrac{\log 0.001}{\log(\frac{2}{3})}$	{Solve for t}
$t \doteq 37.8$	{Evaluate}

The number of germs reduces to less than 0.001 of the original number after 38 days.

5. Given:

$$B = 10\log_{10}\left(\frac{I}{I_0}\right) \qquad \mathbf{1}$$

For B = 100,

$100 = 10\log_{10}\left(\frac{I}{I_0}\right)$	{Substitute}
$10 = \log_{10}\left(\frac{I}{I_0}\right)$	{Divide by 10}
$10^{10} = \dfrac{I}{I_0}$	{Write as an index expression}
$I = 10^{10} I_0$	{Solve for I}

For 2 speakers

$I = 2\times10^{10} I_0$ {2 speakers = 2 x the intensity}

Hence

$B = 10\log_{10}\left(\dfrac{2\times10^{10} I_0}{I_0}\right)$ {Substitute into **1**}

$= 10\log_{10}\left(2\times10^{10}\right)$ {Divide out I_0}

$\doteq 103$ {Evaluate}

The intensity of 2 speakers is only 103 dB.

6. a. Let the current reserves of oil equal 100 units. If the analysts say that the oil will last for 100 years at the current rate of consumption, then they are assuming that we use 1 unit each year.

But if consumption is increasing at 4% per year, we have this model instead:

Year	Oil Used	Total Oil Used
1	1	1
2	1.04	1 + 1.04 = 2.04
3	1.04^2	$1 + 1.04 + 1.04^2 = 3.1216$
4	1.04^3	$1 + 1.04 + 1.04^2 + 1.04^3 = 4.246464$
etc		

We need to determine how many years are needed for the total oil used to exceed 100.
In the next chapter you will learn how to do this algebraically. For now, we will use a spreadsheet.

The formula in C7 is: C7 = (1.04)^(B6)

The formula in D7 is: D7 = D6 + C7

These formulas are copied down the columns until the Total Used > 100.

Perhaps surprisingly, the "100 year supply of oil" will only last 42 years if consumption increases by 4% per year.

	B	C	D
4	**Year**	**Oil Used**	**Total Used**
5	1	1	1
6	2	1.04	2.04
7	3	1.0816	3.1216
8	4	1.124864	4.246464
9	5	1.169859	5.4163226
10	6	1.216653	6.6329755
11	7	1.265319	7.8982945
…	…	…	…
46	42	4.993061	104.8196

b. The mathematics is the identical, albeit with a 6% increase rather than a 4% increase.

	B	C	D
4	**Year**	**Oil Used**	**Total Used**
5	1	1	1
6	2	1.06	2.06
7	3	1.1236	3.1836
8	4	1.191016	4.374616
9	5	1.262477	5.637093
10	6	1.338226	6.975319
11	7	1.418519	8.393838
…	…	…	…
75	71	59.07593	1027.008

Note: This spreadsheet is available from the mathematics-for-queensland.com website.

It only takes 71 years to use up the "1000 year supply of oil" if oil consumption increases by 6% per year!

7. We first need to find a mathematical model for the amount A of space junk being added each year t from 1991 to 2000. Taking 1990 to be year 0, the weight of space junk when $t = 0$ is 1.8 million kg.

The amount of space junk added each year is assumed to be an exponential function that passes through (1, 0.8) and (10, 1.2).

Using ExpReg on a TI-83, we find that the equation is given by: $A = 0.7648 \times 1.046^t$.

One way model this problem is with a spreadsheet, as a spreadsheet is an effective way to calculate the weight of space junk added and the total amount of space junk each year.

The formula in B6 is: B6 = 0.7648 x 1.046^A6

The formula in C6 is: C6 = C5+B6

These formulas are copied down the columns until the Year = 15 (i.e. 2005).

	A	B	C
4	**Year**	**Additional**	**Total**
5	0	0	1.8
6	1	0.800	2.600
7	2	0.837	3.437
8	3	0.875	4.312
9	4	0.916	5.228
10	5	0.958	6.185
11	6	1.002	7.187
12	7	1.048	8.235
13	8	1.096	9.331
14	9	1.146	10.477
15	10	1.199	11.676
16	11	1.254	12.930
17	12	1.312	14.242
18	13	1.372	15.615
19	14	1.435	17.050
20	15	1.501	18.552

Note: This spreadsheet is available from the mathematics-for-queensland.com website.

a. A mathematical model can be an equation, a graph or a table. In this problem, the table given above is our model.

b. The model shows that there will be 18.6 million kg of space junk in 2005.

c. The model shows that there was over 5 million kg of space junk in 1994.

8. Here "solve for *n*" means to find an algebraic expression for *n* in terms of *a*, *b* and *m*.

$a^n = b^{n+m}$	
$\log a^n = \log b^{n+m}$	{Take log of both sides}
$n\log a = (n+m)\log b$	{Apply log law 3}
$n\log a = n\log b + m\log b$	{Expand}
$n\log a - n\log b = m\log b$	{Gather terms with *n* together}
$n(\log a - \log b) = m\log b$	{Factorise}
$n = \dfrac{m\log b}{\log a - \log b}$	{Solve for *n*}