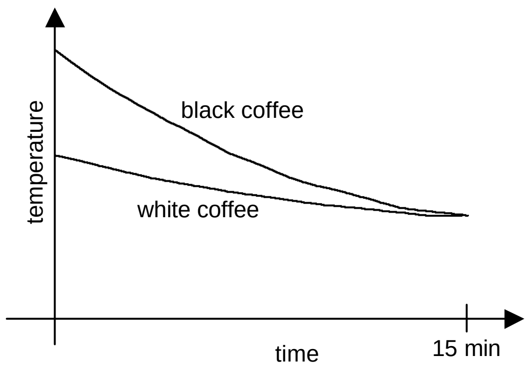
Mathematics for Queensland, Year 12
Worked Solutions to Exercise 8K

1. Darren's coffee is hotter at first (the milk has cooled his wife's cup of coffee), but cools more quickly (since his coffee is in an cup and his wife's is in a thermos. After 15 minutes, both coffees are at the same temperature. The graph will look similar to this.



2. (a) Differentiating R to find its rate of change

$$\frac{dR}{dt} = R_0 k e^{kt} \text{ and this equals } kR_0 e^{kt} \quad \text{thus} \quad \frac{dR}{dt} = kR$$

So the rate of change of R is proportional to R.

(b) Substituting in the formula for R we have $R_0 = 21$ and that $R = 28$ at $t = 34$, thus solving for k we have

$$28 = 21e^{34k} \quad \text{thus} \quad \frac{28}{21} = e^{34k}$$

$$\text{so} \quad e^{34k} = 4/3 \quad \text{thus} \quad 34k = \ln(4/3),$$

$$\text{and finally} \quad k = \frac{1}{34}\ln(4/3) = .00846$$

(c) We need $\frac{dR}{dt} > 1$. Since $\frac{dR}{dt} = kR$ this will happen when $kR > 1$, that is when $.00846R > 1$. Dividing each side by .00846,

$R > \frac{1}{.00846}$, i.e. when $R > 118.18$

<u>So when R is great than 118.18 mm^2 the rust grows at more than 1 mm^2 per day.</u>

3. Since the area under the graph of the function equals the definite integral of the function over the domain of values of the function we have to find A so that the integral of the function, from 0 to 10 equals 1. In terms of calculus we require the value of A so that $\int_0^{10} Ae^{-0.1x}dx = 1$.

Doing the integration

$$\int_0^{10} Ae^{-0.1x}dx = \left[A\frac{1}{-0.1}e^{-0.1x}\right]_0^{10} = A\left[\frac{1}{-0.1}e^{-0.1x}\right]_0^{10} = \frac{A}{(-0.1)}\left[e^{-(0.1)(10)} - 1\right] = 10A(1-e^{-1})$$

So we need $10A(1-e^{-1}) = 1$, thus $A = \dfrac{1}{10(1-e^{-1})} \simeq .1582$

4. By differentiating the velocity with respect to time we get the acceleration. If we do this we will have an expression relating V and t and another relating A, the acceleration and t. By eliminating t between the two expressions we will get a relation between A and V and we can find one of these variables in terms for the other.

Carrying out these steps, $\dfrac{dV}{dt} = 80(-(-0.4e^{-0.4t})) = 32e^{-0.4t} = A$.

From this we have $e^{-0.4t} = \dfrac{A}{32}$, so substituting for $e^{-0.4t}$ in the expression for V we finally have $V = 80(1-\dfrac{A}{32})$. This tidies up to give $V = \dfrac{5}{2}(32-A)$

5. The rate of change of information will be $\dfrac{dI}{dt}$. Note that the expression for I can be written as $I = 3.2\ln(a + 3t)$, which is better for calculus purposes.

Differentiating I, $\dfrac{dI}{dt} = 3(3.2)\dfrac{1}{(a+3t)} = \dfrac{9.6}{(a+3t)}$, we used the composite function rule on the function. This accounts for the term 3 in the middle expression above.

Now as t increases the bracket $(a + 3t)$ increases and so the expression $\dfrac{9.6}{(a+3t)}$ decreases.

So we now realize that $\dfrac{dI}{dt}$ decreases as t increases. It is always positive.

Thus we conclude that I always increases with t, but that it's rate of growth slows with time getting ever closer to zero. <u>Thus the statement in line 5 of the question is true.</u>

6. Using the index laws, $\dfrac{(e^{3x}-e^{-x})}{e^{-2x}} = \dfrac{e^{3x}}{e^{-2x}} - \dfrac{e^{-x}}{e^{-2x}} = e^{3x}e^{2x} - e^{-x}e^{2x} = e^{5x} - e^{x}$ and differentiating the last expression we get $\dfrac{d}{dx}\left(\dfrac{e^{3x}-e^{-x}}{e^{-2x}}\right) = \dfrac{d}{dx}(e^{5x}-e^{x}) = 5e^{5x} - e^{x}$ as required.

We use $_0e^{kt}$. t in billions of years, then when = 1.3, P P_0

So $0.5P_0 = P_0e^{-k(1.3)}$, and after canceling P_0, $e^{-1.3k} = 0.5$

or we can write $e^{-k} = (0.5)^{(1/1.3)}$. Because $(e^{-kt}) = (e^{-k})^t$, this means

$P = P_0((0.5)^{(1/1.3)})^t$

Thus when the ratio is 20% we have $0.2P_0=P_0((0.5)^{1/1.3})^t$. Cancelling P_0, we get

$0.2 = (0.5)^{\frac{t}{1.3}}$ and we solve this for t by taking logarithms of each side to get

$\ln(0.2) = \ln((0.5)^{(t/1.3)})$. Simplifying we get $\ln(0.2) = \frac{t}{1.3}\ln(0.5)$.

This gives $t = \frac{(1.3)\ln(0.2)}{\ln(0.5)} \simeq 3.02$.

<u>So the age of the material is about 3.02 billion years</u>.