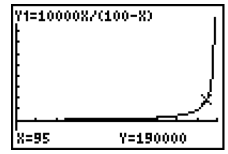
Mathematics for Queensland, Year 11 Mathematics B, A Graphics Calculator Approach
http://mathematics-for-queensland.com

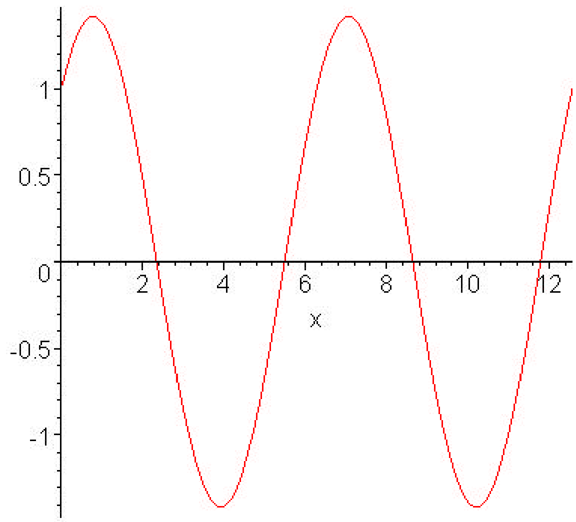
Exercise 9I – Worked Solutions to the Exercise

1. For this problem we can get an idea of the answer by plotting the function.



We know that the function sin and cos repeat themselves in an interval of length 2π so if we take a range from 0 to 4π this should give us a good look at the function.

We see that the maximum and minimum values are about 1.4.



We will find the places where $\frac{dy}{dx}$ is zero and then test for maxima and minima.

So $\frac{dy}{dx} = \cos x - \sin x$ and this is zero when $\cos x - \sin x = 0$, i.e. when $\cos x = \sin x$

Now this is solved when $\frac{\sin x}{\cos x} = 1$, and since $\frac{\sin x}{\cos x} = \tan x$ we have $\tan x = 1$

Remember from your list of values of sin, cos and tan at angles of $\frac{p}{6}, \frac{p}{4}, \ \ldots$, that $\tan\frac{p}{4} = 1$,

so $x = \pi/4$ is one place where $\frac{dy}{dx} = 0$. This is not the only place, for we also know that

$\tan\frac{5p}{4} = 1$, so $x = 5\pi/4$ is a second place.

By considering the sign of $\frac{dy}{dx}$ just before and just after we see that $\frac{dy}{dx}$ changes from positive to negative so there is a local maximum at $x = \pi/4$ ($\simeq .78$)

Considering the sign of $\frac{dy}{dx}$ just before and just after $x = 5\pi/4$ ($\simeq 3.93$)we see that $\frac{dy}{dx}$ changes from negative to positive so there is a local minimum at $x = 5\pi/4$.

(Or apply the second derivative test at these two points).

At $x = \pi/4$, $y = \cos(\frac{p}{4}) + \sin(\frac{p}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \simeq 1.4$.

Similarly, at $x = 5\pi/4$, $y = \cos(\frac{5p}{4}) + \sin(\frac{5p}{4}) = -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}} = -\sqrt{2} \simeq -1.4$

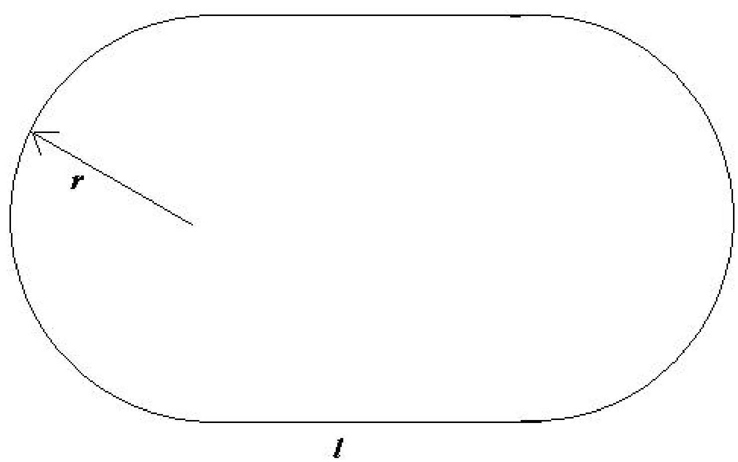
These results are consistent with the graph above.

These values repeat themselves in intervals of length 2π thus the maximum and minimum values of $y = \cos x + \sin x$ are $\sqrt{2}$ and $-\sqrt{2}$.

2. We need a formula for the area of the playing field. So we divide the area into parts whose areas we can find. We have a rectangle and two semicircular parts, so in fact we have a rectangle plus a circle.

   It is clear that the radii of the semicircular parts are half one side of the rectangle, so if we know the sides of the rectangle we can find the areas of each piece. The requirement that the running track has length 400 metres means that there is a relation between the radii of the semicircles and the straight parts. This relation can be used to eliminate one of the variables in the expression for area and from that we can use calculus to find the greatest area.

   Referring to the diagram



Area of the two semicircular pieces = πr^2 . Area of the rectangle = $2rl$.

Thus the total area A enclosed by the track is given by $A = \pi r^2 + 2rl$.

Similarly the length of the two semicircular arcs is $2\pi r$ and each side has $2l$

Thus the total perimeter = $2\pi r + 2l$.

Since the running track is to be 400 metres, we have $2\pi r + 2l = 400$, rearranging gives $l = 200 - \pi r$.

Thus $A = \pi r^2 + 2r(200 - \pi r) = 400r - \pi r^2$.

Differentiating A with respect to r we have $\frac{dA}{dr} = 400 - 2p\,r$. Thus

$\frac{dA}{dr} = 0$ when $400 - 2p\,r = 0$, i.e. when $r = \frac{200}{p}$.

As r increases through $200/\pi$ $\dfrac{dA}{dr}$ changes from positive to negative so this stationary point for A is a local maximum. (You could apply the second derivative test here instead). Since it is the only stationary point it must be the absolute maximum.

The area enclosed is a maximum when the radius equals $200/\pi$ metres, i.e. approximately 63.7 metres

The strange thing here is that the length of the straight parts is zero. There is a simple explanation for this . Ask your maths teacher why. We could have written down the answer from the start!

3. We find the derivative of C with respect to t and find where it is zero, then we test for the nature of any of the stationary points.

$$\frac{dC}{dt} = \frac{k(-be^{-bt} - (-ae^{-at}))}{b-a} = \frac{-k(be^{-bt} - ae^{-at})}{b-a}$$

This is zero when the bracket $(be^{-bt} - ae^{-at})$ is zero. So we have $be^{-bt} = ae^{-at}$. Thus

$$\frac{e^{-bt}}{e^{-at}} = \frac{a}{b}$$

so $e^{(-bt-(-at))} = e^{(a-b)t} = \dfrac{a}{b}$ and so $(a-b)t = \ln(a/b)$, i.e. $t = \dfrac{\ln(a/b)}{(a-b)}$

This is the only point where $\dfrac{dC}{dt} = 0$ and frrm the expression for C, at $t = 0$, $C = 0$ and also C goes to zero as t goes to infinity. Because of the conditions on k, a, and b, the value of C is positive. Thus the place where $\dfrac{dC}{dt} = 0$ is a local maximum and so the greatest value of C is at $t = \dfrac{\ln(a/b)}{(a-b)}$.

4. We use the formula that $time\ taken = \dfrac{distance}{velocity}$. Referring to the diagram on page 299, since BP = x, from the right angled triangle ABP we deduce that $AP^2 = AB^2 + BP^2$ thus $AP^2 = 9 + x^2$ and $AP = \sqrt{(9+x^2)}$

Since Jimmy rows from A to P at 4 km/h. , his time taken from A to P is $\dfrac{\sqrt{9+x^2}}{4}$.

His time running from from P to C is $\dfrac{5-x}{8}$ since he runs $5-x$ km at 8 km/h.

Let T be his total time, then $T = \dfrac{\sqrt{9+x^2}}{4} + \dfrac{5-x}{8} = \dfrac{1}{4}\sqrt{9+x^2} + \dfrac{5}{8} - \dfrac{x}{8}$ (these times are in hours because his speed is in km/h. and the distances are kilometres)

Calculating $\dfrac{dT}{dx}$ we get $\dfrac{dT}{dx} = \dfrac{1}{4}(2x)(\dfrac{1}{2}(9+x^2)^{-1/2}) - \dfrac{1}{8}$. This simplifies to

$$\frac{dT}{dx} = \frac{x}{4(9+x^2)^{1/2}} - \frac{1}{8}$$

So $\dfrac{dT}{dx} = 0$ if $\dfrac{x}{4(9+x^2)^{1/2}} = \dfrac{1}{8}$, i.e., if

$\dfrac{x}{(9+x^2)^{1/2}} = \dfrac{4}{8} = \dfrac{1}{2}$ and squaring both sides $\dfrac{x^2}{9+x^2} = \dfrac{1}{4}$

Cross multiplying we have $4x^2 = 9 + x^2$ so $3x^2 = 9$, $x^2 = 3$, so $x = \sqrt{3}$.

The value of $\dfrac{dT}{dx}$ at $x = 0$ is $-1/8$ and at $x = 5$ it is $\dfrac{\sqrt{34}}{4}$. The important thing is that $\dfrac{dT}{dx}$ is zero at only one place, and that it changes from −ve to +ve at $x = \sqrt{3}$.

So T must be a minimum at this value and its value is

$$\frac{1}{4}\sqrt{12} + \frac{5}{8} - \frac{\sqrt{3}}{8} = \frac{1}{4}\sqrt{4}\sqrt{3} + \frac{5}{8} - \frac{\sqrt{3}}{8} = \frac{3}{8}\sqrt{3} + \frac{5}{8} \simeq 1.25 \text{ hours}$$

(b) The answer in the textbook is not quite complete. If we replace the velocity when he rows by b and apply calculus again, we'll get $x = \dfrac{3b}{\sqrt{64-b^2}}$, but note that if b 8 then the expression for x contains the square root of a negative number. Thus the formula for x is only valid for $b < 8$.

What happens if $b \geq 8$? Think about the problem it means that Jimmy can row faster than he can run, extremely unlikely, unless the river was flowing very quickly. In that unlikely event he should row all the way and the solution is $x = 5$.

In fact $x = 5$ is the solution for all values of $b \geq \dfrac{40}{\sqrt{34}} \simeq 6.86$. This is true because if

$b > \dfrac{40}{\sqrt{34}}$ then the local minimum is beyond $x = 5$ and the time is a decreasing function of x for the whole interval from 0 to 5. To verify this plot some graphs for a range of values of b.

5. The diagram on page 450 in the text appears in part (a) of question 4 is the diagram for this question. The hint then shows that the area is given by $A = 2x\sqrt{4-x^2}$. It is important to note that x must lie between 0 and 2.

Differentiating A with respect to x we have a product with $u(x) = x$ and $v(x) = \sqrt{4-x^2}$.
The derivative of $v(x)$ has to found by the composite function rule.

So we put $z = x^2$ then $v = \sqrt{4-z}$ then $\dfrac{dv}{dx} = \dfrac{dv}{dz}\dfrac{dz}{dx} = -\dfrac{1}{2}\dfrac{1}{\sqrt{4-z}}2x$. So

$\dfrac{dv}{dx} = -\dfrac{x}{\sqrt{4-x^2}}$.

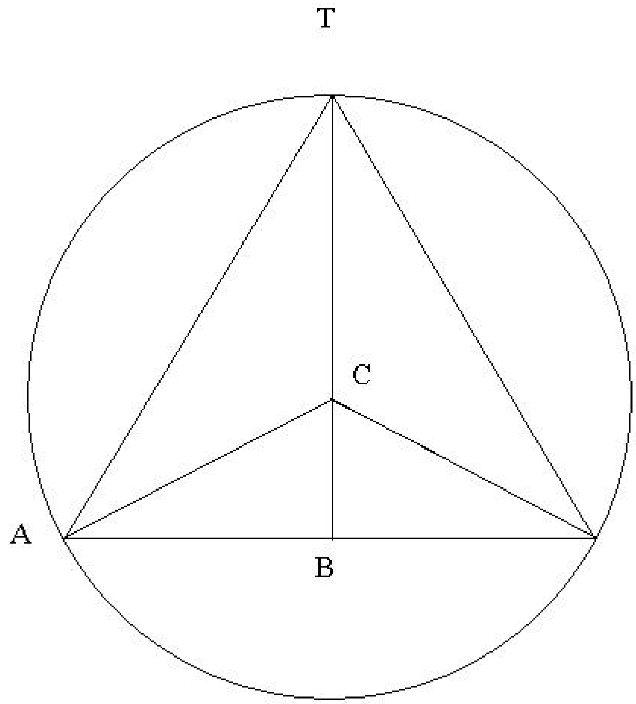
Now applying the product rule we have $\dfrac{dA}{dx} = -\dfrac{x}{\sqrt{4-x^2}}x + \sqrt{4-x^2}$.

Simplifying this expression, $\dfrac{dA}{dx} = \dfrac{-x^2 + (4-x^2)}{\sqrt{4-x^2}} = \dfrac{2(2-x^2)}{\sqrt{4-x^2}}$. On the interval $0 \leq x \leq 2$,

$\dfrac{dA}{dx} = 0$ at $x = \sqrt{2}$.

Since $A = 0$ at $x = 0$ and $x = 2$ and there is only one stationary point in the interval it is clear that A must be a maximum at $x = \sqrt{2}$.

6. We have to find the volume of a cone which fits into a sphere. We could use the radius of the base of the cone, the height of the cone or the semivertical angle of the cone. The choice will affect the manipulations, some will be considerably more complicated than others.



AC = r, AB = x, CB = y, TB = h.

α = the semivertical angle ATB, 2α = angle ACB.

With these notations using the geometry of the figure we have BC = $r\cos 2\alpha$, AB = $r\sin 2\alpha$, $h = r + y$, $h = r + r\cos 2\alpha$, $x^2 + y^2 = r^2$, $y = \sqrt{r^2 - x^2}$.

Using the formula $V = \frac{1}{3}p\,r^2 h$, we have (i) with α as the variable

$V = \frac{1}{3}r^2 \sin 2a\,(r + r\cos 2a)$

(ii) with h as the variable $V = \frac{1}{3}p\,h(r^2 - (h-r)^2)$ (iii) with x as variable

$V = \frac{1}{3}p\,x^2(\sqrt{r^2 - x^2} + r)$

(iv) with y as the variable $V = \frac{1}{3}p\,(r^2 - y^2)(r + y)$. Note that we can have y negative here, the range of y is $-r \le y \le r$

While some simplification is possible with each of the expressions above we shall use the fourth one. Then $\frac{dV}{dy} = \frac{1}{3}p\left(1(r^2 - y^2) + (r+y)(-2y)\right) = \frac{1}{3}p\left(r^2 - 2ry - 3y^2\right)$.

The bracket factorises into $(r + y)(r - 3y)$ so $\dfrac{dV}{dy}$ is zero at $y = r/3$ and V is zero at $y = r$ and $y = -r$.

Since $y = r/3$ is the only stationary point inside the interval it must give the greatest volume there.

Substituting this value into V we find that the maximum volume is

$$\frac{1}{3}p\,(r^2 - \frac{r^2}{9})(r + \frac{r}{3}) = p\,\frac{32r^3}{81}.$$

7. Referring to the diagram in the textbook we can use x or y as the variable here.

The two expressions for the volume are, using *volume* = (*height*)(*base area*),

$V = 2yp\,(r^2 - y^2)$ and $V = 2\sqrt{r^2 - x^2}p\,x^2$, both x and y lie between 0 and r.

It is easy to differentiate the expression for V in terms of y with respect to y either by the product rule or expanding out the expression first. In either case there is only one stationary point and since V is zero at the ends of the range the point is a local maximum. This is therefore is the global maximum also. It is taken at $y = r/\sqrt{3}$ and so the maximum volume is $\dfrac{4p\,r^3}{3\sqrt{3}}$.

8. You can do the problem by expressing the dimensions of the cone using its semivertical angle, but the problem comes out easily using either the height of the cylinder or the radius. As stated in the hint one can use similar triangles to relate r the base radius of the cylinder and its height h.

There are three similar triangles. The large cross section of the cone, a smaller one with one side of length h and a third with a side r.

Applying the similarity relations for the large triangle and that having side h we have

$$\frac{h}{R - r} = \frac{H}{R}$$

Applying the similarity relations for the large triangle and that having side r we have

$$\frac{r}{H - h} = \frac{R}{H}$$

Using the first of these we have Volume of the cylinder ,

$$V = p\,r^2h = p\,r^2\,\frac{H(R - r)}{R} = p\,\frac{H}{R}(Rr^2 - r^3)$$

Differentiating with respect to r we get $\dfrac{dV}{dr} = p\,\dfrac{H}{R}\left(2Rr - 3r^2\right)$. This is zero at $r = 0$ and at $r = 2R/3$.

That this is the point at which maximum volume is taken follows from the same arguments used in questions 5, 6, and 7.

We find that the maximum volume equals $\dfrac{4p}{27}HR^2$.