Mathematics for Queensland, Year 12
Worked Solutions to Exercise 9L

1. The information in the question tells us that the derivative of this function will be zero at $x = -1$.

   Differentiating the expression for the cubic, we have $f'(x) = 3x^2 + 2ax$ so since

   $f'(-1) = 0$

   we can get an equation for the unknown constant a.

   Thus $3(-1)^2 + 2a(-1) = 0$.

   Rearranging this expression we find $\underline{a = 3/2}$.

2. Setting up this problem is the challenge. We have to find an expression for the yield per hectare.

   Note that we won't need to know the area of the orchard, since for an orchard of any size we would multiply the yield per hectare by the area.

   It helps, perhaps, to introduce an area A then you know that the number of trees in the area is Ax.

   Then the yield from this orchard will be the number of trees times the yield per tree.

   The yield is $Ax(500-5x)$, thus dividing by the area will give the yield per hectare.

   We therefore find that the yield per hectare, y is given by

   $y = x(500-5x)$.

   In the problem, x can't be negative and if the density of trees were greater than 100 the yield would be negative, thus the problem is:

   Maximize $y = x(500-5x)$ where $0 < x < 100$.

   Using calculus or algebra we find that the maximum value of this function is taken at $x = 50$, because $\frac{dy}{dx} = 100 - 2x$ which is zero at $x = 50$ and is a local maximum point since $\frac{dy}{dx}$ changes from positive to negative at $x = 50$. Being the only stationary point in the given domain, the point must give the greatest value of y. Thus:

   $\underline{\text{The yield is greatest where } x = 50.}$

3. The diagram is a little misleading because it looks as though the base is the interval from 0 to 1, but in fact it is a general point and the base of the area goes from 0 to x.

Since P is the point (x,y) the rectangle has height y and width x, thus it has area xy .

Denote this by R so $R = xy$.

Since P is on the graph of $y = 4 - x^2$ we can substitute for y then we have
$R = x(4 - x^2)$

As shown in the diagram we only take values of x between 0 and 2.
Now $\frac{dR}{dx} = 4 - 3x^2$ and so $\frac{dR}{dx} = 0$ when $x = \sqrt{\frac{4}{3}}$ or $-\sqrt{\frac{4}{3}}$. Equivalently $x = \pm\frac{2}{\sqrt{3}}$
The latter value is not in the domain of the problem.
By considering the sign of $\frac{dR}{dx}$ as x passes through $2/\sqrt{3}$ or evaluating $\frac{d^2R}{dx^2}$ at this point we can show that this local stationary point is a maximum point and being the only stationary point in the domain, it is the place where the area is greatest.

<u>So the area is greatest at $x = 2/\sqrt{3} \cong 1.15$.</u>

4. Let the printed area be x by y cms. The total area will measure $x + 3$ cms. by $y + 4$ cms.

So we have to minimize $A = (x + 3)(y + 4)$. The printed area is to be 35 cms.^{2}, so $xy = 35$.

Eliminating y we have

$$A = (x+3)(y+4) = xy + 3y + 4x + 12 = 35 + 3(\frac{35}{x}) + 4x + 12 = 47 + \frac{105}{x} + 4x$$

So our problem is to find the minimum value of A over all values of $x > 0$.
Differentiating A we have $\frac{dA}{dx} = 4 - \frac{105}{x^2}$. this is zero when $x = \pm\sqrt{(105/4)}$.
The negative value is outside the domain of the problem so there is one stationary point.
From the expression for A , as x increases indefinitely A gets larger and larger.
Likewise, as x gets near zero, A gets larger and larger.
Hence at $x = \sqrt{(105/4)}$ A must have a local minimum and this being the only minimum it must be the global minimum.

<u>Thus the least area of cardboard used is when the length is $\simeq$ 3+5.123 cm and width 4+6.831 cm so that the area is 87.98 sq cm.</u>

5. With the square pieces cut out as shown in the diagram, if the square pieces are x by x cm. the height (or depth) of the box will be x cms.

The length of the cardboard consists of the top , the bottom and four widths of size x cms. thus the top and bottom pieces will use length $95 - 4x$ cms.

Since the top and bottom are the same dimensions the length of these two pieces is $(95\text{-}4x)/2$ cms.

The width of the top and bottom will be 47 cms less the amount used for the two corner pieces.

So the width of the top and bottom will $47\text{-}2x$ cms.

Thus the box will be a shallow rectangular volume measuring x, by $(47\text{-}2x)$, by $(95\text{-}4x)/2$ cms

So the volume will be $V = x(47\text{-}2x)(95\text{-}4x)/2$ cu.cms.

Because each of the sides have to have a length greater than zero x has to be such that $x > 0$, $(95 - 4x)/2 > 0$ and $47\text{-}2x > 0$.

The last two inequalities can then be rearranged to give, $x < 95/4$ (= 23.75) and $x < 74/2$(= 23.5) Since we require both of these to be true the values of x must satisfy $x < 23.5$

Thus the problem becomes :

Maximise $V = x(47\text{-}2x)(95\text{-}4x)/2$ for $0 < x < 23.5$

We find $\dfrac{dV}{dx} = 4465 - 756x + 24x^2$. This is zero at $x \simeq 7.875$ and 23.625.

The second value is outside the domain. At the first point the value of $\dfrac{d^2V}{dx^2} = 756 - 48(7.875) = 378$, positive, so the first is a maximum point and is the only value at which $\dfrac{dV}{dx}$ is zero thus it is the value at which value at which the volume is greatest.

6. THIS IS A HARD PROBLEM.

(a) The domain is $x > 0$ and $x \neq 1$, since ln *x* is only defined for positive values, but is 0 at $x = 1$.

(b) By the quotient rule $f\,'(x) = \dfrac{\ln x - x(\frac{1}{x})}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$ and so is 0 when $\ln x = 1$, i.e., $x = e$.

(c) $\ln x > 1$ for $x > e$ so $f'(x)$ is positive for $x > e$. Thus *f* is increasing for for $x > e$.

(d) By evaluating the function for *x* near zero it can be seen that *f*(*x*) goes to 0 as *x* goes to zero.

The function goes from 0 to $-\infty$ as *x* goes from 0 to 1.

For $x > 1$ and near 1, *f*(*x*) is large and positive, because ln *x* is near zero.

It also goes to ∞ as *x* goes to ¥ , you can get an idea of this by evaluating the function for an increasing set of values.

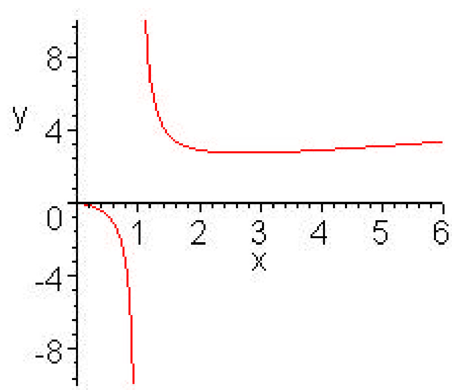
In fact it appears that the ln *x* term in the denominator would make *f* go to zero, while the term *x* would make it get increasingly larger. In this situation it is the power which dominates and *f*(*x*) goes to ∞ as *x* goes to ∞.

This means that the point $x = e$ is the place where *f*(*x*) is least on the range from 1 to ∞.

So for $x > 1$ the range of *x* values is from $e/\ln(e) = e$ to ∞ .

The range of values from 0 to 1 is $-\infty$ to 0, so the complete range of values is the union of the sets $-\infty < x < 0$ and $e \leq x < \infty$

(e) Here's a Maple plot of the function.



7. Let the foot of the wall be C. ABC is a right-angled triangle. Let the top and bottom of the fence be D and E. Then ADE is a right-angled triangle , similar to ABC.

Let length of BC = y and length of AE = x, then length of AC = x+1, and length of DE = 2

Then by similar triangles,

$$\frac{AC}{AE} = \frac{BC}{DE} \quad \text{so} \quad \frac{x+1}{x} = \frac{y}{2}.$$

By Pythagoras' Theorem, $AB^2 = (x + 1)^2 + y^2 = (x+1)^2 + \frac{4(x+1)^2}{x^2} = (x+1)^2\left(1+\frac{4}{x^2}\right)$

If we find when AB^2 is least, this will determine when AB is least.

Let $AB^2 = z$, then by the product rule

$$\frac{dz}{dx} = 2(x+1)(1+\frac{4}{x^2}) + (x+1)^2(-\frac{8}{x^3}) = 2(x+1)\left(1+\frac{4}{x^2}-\frac{4}{x^3}(x+1)\right)$$

Simplifying the large bracket leads to

$$\frac{dz}{dx} = 2(x+1)\left(1+\frac{4}{x^2}-\frac{4}{x^2}-\frac{4}{x^3}\right) = 2(x+1)\left(1-\frac{4}{x^3}\right)$$

In the problem x has to be non-negative, so $\frac{dz}{dx}$ can be zero only when

$$1-\frac{4}{x^3} = 0, \quad \text{i.e. when } x = 4^{\frac{1}{3}}.$$

Now as x gets near zero z gets large due to the term $4/x^2$, while as x gets large z gets large due to the term $(x + 1)^2$. Since there is only one stationary point for $x > 0$ the point must be a minimum. So the minimum is taken at $x = 4^{1/3} \simeq 1.587$ and then

$$z = AB^2 = (1+4^{1/3})^2\left(1+\frac{4}{4^{2/3}}\right) = (1+4^{(1/3)})^3 \simeq 17.321 \text{ and so finally,}$$

Length of the beam, AB = $(1+4^{(1/3)})^{3/2} \simeq 4.162$ metres